SIMULTANEOUS INFERENCE FOR TIME-VARYING MODELS

By Sayar Karmakar[1], Stefan Richter[2] and Wei Biao Wu[3]

University of Florida, Heidelberg University and University of Chicago

[1]Corresponding Author:
Department of Statistics
University of Florida
230 Newell Drive
Gainesville, FL 32611, USA.
E-mail: sayarkarmakar@ufl.edu

,

[2]Institut für Angewandte Mathematik
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg, Germany.
E-mail: stefan.richter@iwr.uni-heidelberg.de

,

[3]Department of Statistics
University of Chicago
5747 S. Ellis Avenue
Chicago, IL 60637, USA.
E-mail: wbwu@galton.uchicago.edu

A general class of time-varying regression models which cover general linear models as well as time series models is considered. We estimate the regression coefficients by using local linear M-estimation. For these estimators, weak Bahadur representations are obtained and are used to construct simultaneous confidence bands. For practical implementation, we propose a bootstrap based method to circumvent the slow logarithmic convergence of the theoretical simultaneous bands. Our results substantially generalize and unify the treatments for several time-varying regression and auto-regression models. The performance for ARCH and GARCH models is studied in simulations and a few real-life applications of our study are presented through analysis of some popular financial datasets.

1. Introduction. Time-varying dynamical systems have been studied extensively in the literature of statistics, economics and related fields. For stochastic processes observed over a long time horizon, stationarity is often an over-simplified assumption that ignores systematic deviations of parameters from constancy. For example, in the context of financial datasets, empirical evidence shows that external factors such as war, terrorist attacks, economic crisis, some political event etc. introduce such parameter inconstancy. As Bai [3]

1

points out, 'failure to take into account parameter changes, given their presence, may lead to incorrect policy implications and predictions'. Thus functional estimation of unknown parameter curves using time-varying models has become an important research topic recently. In this paper, we propose a general setting for simultaneous inference of local linear M-estimators in semi-parametric time-varying models. Our formulation is general enough to allow unifying time-varying models from the usual linear regression, generalized regression and several auto-regression type models together. Before discussing our new contributions in this paper, we provide a brief overview of some previous works in these areas.

In the regression context, time-varying models are discussed over the past two decades to describe non-constant relationships between the response and the predictors; see, for instance, Fan and Zhang [17], Fan and Zhang [18], Hoover et al. [24], Huang, Wu and Zhou [25], Lin and Ying [35], Ramsay and Silverman [42], Zhang, Lee and Song [54] among others. Consider the following two regression models

$$\text{Model I: } y_i = x_i^\mathsf{T}\theta_i + e_i, \quad \text{Model II: } y_i = x_i^\mathsf{T}\theta_0 + e_i, \qquad i = 1, \ldots, n,$$

where $x_i \in \mathbb{R}^d$ $(i = 1, \ldots, n)$ are the covariates, T is the transpose, θ_0 and $\theta_i = \theta(i/n)$ are the regression coefficients. Here, θ_0 is a constant parameter and $\theta : [0, 1] \to \mathbb{R}^d$ is a smooth function. Estimation of $\theta(\cdot)$ has been considered by Hoover et al. [24], Cai [8]) and Zhou and Wu [59] among others. Hypothesis testing is widely used to choose between model I and model II, see, for instance, Zhang and Wu [55], Zhang and Wu [56], Chow [10], Brown, Durbin and Evans [6], Nabeya and Tanaka [38], Leybourne and McCabe [32], Nyblom [39], Ploberger, Krämer and Kontrus [41], Andrews [2] and Lin and Teräsvirta [33]. Zhou and Wu [59] discussed obtaining simultaneous confidence bands (SCB) in model I, i.e. with additive errors. However their treatment is heavily based on the closed-form solution and it does not extend to processes defined by a more general recursion. Little has been known for time-varying models in this direction previously.

The results from time-varying linear regression can be naturally extended to time-varying AR, MA or ARMA processes. However, such an extension is not obvious for conditional heteroscedastic (CH) models. These are difficult to estimate but also often more useful in analyzing and predicting financial datasets. Since Engle [15] introduced the classical ARCH model and Bollerslev [5] extended it to a more general GARCH model, these have remained primary tools for analyzing and forecasting certain trends for stock market datasets. As the market is vulnerable to frequent changes, non-uniformity across time is a natural phenomenon. The necessity of extending these classical models to a set-up where the parameters can change across time has been pointed out in several references; for example Stărică and Granger [47], Engle and Rangel [16] and Fryzlewicz, Sapatinas and Subba Rao [20]. Towards time-varying parameter models in the CH setting, numerous works discussed the CUSUM-type procedure, for instance, Kim, Cho and Lee [29] for testing change in parameters of GARCH(1,1). Kulperger et al. [31] studied the high moment partial sum process based on residuals and applied it to residual CUSUM tests in GARCH models. Interested readers can find some more changepoint detection results in the context of CH models in

James Chu [26], Chen and Gupta [9], Lin et al. [34], Kokoszka et al. [30] or Andreou and Ghysels [1].

Historically in the analysis of financial datasets, the common practice to account for the time-varying nature of the parameter curves was to transfer a stationary tool/method in some ad hoc way. For example, in Mikosch and Stărică [37], the authors analyzed S&P500 data from 1953-1990 and suggested that time-varying parameters are more suitable due to such a long time-horizon. They re-estimated the parameters for every block of 100 sample points and to account for the abrupt fluctuation of the coefficients, they generated re-estimates of parameters for samples of size $100, 200, \ldots$. This treatment suffers from different degree of reliability of the estimators at different parts of the time horizon. There are examples outside the analysis of economic datasets, where similar approach of splitting the time-horizon has been adapted to fit CH type models. For example, in Giacometti et al. [22], the authors analyzed Italian mortality rates from 1960-2003 using an AR(1)-ARCH(1) model and observed abrupt behavior of yearwise coefficients. Our framework can simultaneously capture these models and provide significant improvements over such heuristic treatments.

A time-varying framework and a pointwise curve estimation using M-estimators for locally stationary ARCH models was provided by Dahlhaus and Subba Rao [14]. Since then, while several pointwise approaches were discussed in the tvARMA and tvARCH case (cf. Dahlhaus and Polonik [12], Dahlhaus and Subba Rao [14], Fryzlewicz, Sapatinas and Subba Rao [20]), pointwise theoretical results for estimation in tvGARCH processes were discussed in Rohan and Ramanathan [45] and Rohan [44] for GARCH(1,1) and GARCH(p,q) models. Even though the conditional heteroscedastic model remained widely popular in analyzing many different types of econometric data, the topic of simultaneous inference in this field remains relatively untouched. When these datasets are observed over a long horizon of time, it is only natural that the parameters will also vary smoothly over time but one cannot infer about the general pattern of these parameter functions (of time) from the pointwise bands. Consider the simple tvARCH(1) model

$$X_i = \sigma_i \zeta_i, \zeta_i \sim N(0,1), \sigma_i^2 = \alpha_0(i/n) + \alpha_1(i/n)y_{i-1}^2.$$

Unlike pointwise confidence bands, we construct of simultaneous confidence bands cover the parameters $\alpha_0(i/n)$ and $\alpha_1(i/n)$ over the whole interval (0; 1]. This allows us to make valid interpretation of confidence intervals that can be compared across time. For example, if the lower end of the $(1 - \alpha)\%$ interval for $\alpha_0(\cdot)$ at $t = 0.2$ is higher than the upper end of the same at $t = 0.8$ then this is indicative of evidence to conclude that the parameter varies as functions of time. This conclusion cannot be validly drawn if we were to construct pointwise confidence intervals. However, the literature on simultaneous intervals is sparse in the time-varying literature and to the best of our knowledge did not exist before this work. While one remedy could be to subjectively assume a certain class of functions such as linear or polynomial and perform a hypothesis test, this can be problematic for many real life datasets. See for example the intercept function for the USGBP analysis in Section

5. We rather take an objective approach where we do *not* assume any parametric form as such and wish to establish valid simultaneous inference. Post our estimation of the function and construction of the simultaneous bands, one can perform many relatively subjective hypothesis such as time-constancy, linearity etc. at one go.

We next summarize our contributions in this paper. We provide a unifying framework that binds linear regression models, generalized regression models and many popularly used auto-regressive models including CH type processes. Moreover, we use Bahadur representations, a Gaussian approximation theorem from Zhou and Wu [58] and extreme value Gaussian theory to obtain SCBs for contrasts of the parameter curves. These intervals provide a generalization from testing parameter constancy to testing any particular parametric form such as linear, quadratic, exponential etc. To deal with bias expansions, we use the theory about derivative processes which was recently formalized in Dahlhaus, Richter and Wu [13].

Moving on to some practical applicability of our results, we show how our result applies to time-varying ARCH and GARCH models. [1] For the GARCH, we improve the existing condition to make it eligible for our theory of SCB. Compared to 8 moments now we need only $4 + a$ moments with small a. We provide an empirical justification of how the coverage can be significantly improved by a wild bootstrap technique and use Gaussian approximation theory to theoretically establish it. Finally we also provide some data analysis and volatility forecasting. First we show for numerous real-life datasets that the time-varying fit does better than the time-constant ones in short-range forecasts. Note that, our goal for this paper is simultaneous inference and not forecasting. However one interesting find from our analysis is how the inference can lead to models where a subset is time-varying and these semi-time varying model can sometimes achieve both statistical confidence and better forecasting ability.

The rest of the article is organized as follows. In Section 2, we state two specific class of time-series models and the related assumptions. For the sake of better focus and readability, we decided to narrow down to specific models in Section 2. However our theoretical results of M-estimation are much more general than that appears in the main manuscript. These general results are collated in the Appendix sections 6 and 7 where we talk about our framework, the functional dependence measure, the assumptions etc. Section 3 consists of the results about the Bahadur representation and the SCBs of the related contrasts of parameter functions. Section 4 is dedicated to practical issues which arise when using the SCBs like estimation of the dispersion matrix of the estimator, bandwidth selection and a wild Bootstrap procedure to overcome the slow logarithmic convergence from the theoretical SCB. Some summarized simulation studies and real data applications can be found in Section 5. All the proofs are also deferred to Appendix Section 8.

---

[1] Our general theory from the Appendix sections 6 and 7 can also cover time-varying generalized model (See an earlier version at [27]) but we suppress it to focus more on econometric datasets and the conditional heteroscedastic models.

2. Model assumptions and estimators.

2.1. *The model.* Suppose that ζ_i, $i \in \mathbb{Z}$ is a sequence of i.i.d. random variables. We consider the following two time series models. In both cases, Θ denotes a parameter space specified below in Section 2.3.

- Case 1: Recursively defined time series. Suppose that for $i, \ldots, n$,

(2.1) $$Y_i = \mu(Y_{i-1}, ..., Y_{i-p}, \theta(i/n)) + \sigma(Y_{i-1}, ..., Y_{i-p}, \theta(i/n))\zeta_i,$$

where $\theta = (\alpha_1, \ldots, \alpha_k, \beta_0, \ldots, \beta_l)^{\mathsf{T}} : [0, 1] \to \Theta \subset \mathbb{R}^{k+l+1}$ and

$$\mu(x, \theta) := \sum_{i=1}^{k} \alpha_i m_i(x), \qquad \sigma(x, \theta) := \Big(\sum_{i=0}^{l} \beta_i \nu_i(x) \Big)^{1/2},$$

with some functions $m_i : \mathbb{R}^p \to \mathbb{R}$, $\nu_i : \mathbb{R}^p \to \mathbb{R}_{\geq 0}$. Put $X_i^c = (Y_{i-1}, \ldots, Y_{1 \vee (i-p)}, 0, \ldots)^{\mathsf{T}}$. This model covers, for instance, tvARMA and tvARCH models.
- Case 2: tvGARCH. For $i = 1, \ldots, n$, consider the recursion

$$
\begin{aligned}
Y_i &= \sigma_i^2 \zeta_i^2, \\
\sigma_i^2 &= \alpha_0(i/n) + \sum_{j=1}^{m} \alpha_j(i/n) Y_{i-j} + \sum_{j=1}^{l} \beta_j(i/n) \sigma_{i-j}^2,
\end{aligned}
$$

where $\theta = (\alpha_0, \ldots, \alpha_m, \beta_1, \ldots, \beta_l) : [0, 1] \to \Theta \subset \mathbb{R}^{m+l+1}$. Put $X_i^c := (Y_{i-1}, ..., Y_1, 0, 0, ...)$.

Case 1 does not directly cover the tvGARCH model, we therefore operate with it separately as Case 2 throughout the paper.
In either case, our goal is to estimate $\theta(\cdot)$ from the observations $Z_i^c = (Y_i, X_i^c)$, $i = 1, \ldots, n$.

2.2. *The estimator.* In this paper, we focus on local M-estimation: Let $K(\cdot) \in \mathcal{K}$, where $\mathcal{K}$ is the family of non-negative symmetric kernels with support $[-1, 1]$ which are continuously differentiable on $[-1, 1]$ such that $\int_{-1}^{1} |K'(u)|^2 du > 0$. We consider as objective function $\ell(z, \theta)$ the negative conditional Gaussian likelihood. This reads

- in Case 1:
$$\ell(y, x, \theta) = \frac{1}{2} \Big[\Big(\frac{y - \mu(x, \theta)}{\sigma(x, \theta)} \Big)^2 + \log \sigma(x, \theta)^2 \Big],$$

- in Case 2:
$$\ell(y, x, \theta) = \frac{1}{2} \Big[\frac{y}{\sigma(x, \theta)^2} + \log(\sigma(x, \theta)^2) \Big],$$

where $\sigma(x, \theta)^2$ is recursively defined via $\sigma(x, \theta)^2 = \alpha_0 + \sum_{j=1}^{m} \alpha_j x_j + \sum_{j=1}^{l} \beta_j \sigma(x_{j \to}, \theta)^2$ and $x_{j \to} := (x_{j+1}, x_{j+2}, \ldots)$.

For some bandwidth $b_n > 0$, define the local linear likelihood function

$$(2.2) \qquad L^c_{n,b_n}(t, \theta, \theta') := (nb_n)^{-1} \sum_{i=1}^{n} K_{b_n}(t - i/n)\ell(Z^c_i, \theta + \theta' \cdot (i/n - t)),$$

where $K_{b_n}(\cdot) := K(\cdot/b_n)$. Let $\Theta' := [-R, R]^k$ with some $R > 0$. A local linear estimator of $\theta(t)$, $\theta'(t)$ is given by

$$(2.3) \qquad (\hat{\theta}_{b_n}(t), \widehat{\theta'}_{b_n}(t)) = \operatorname*{argmin}_{(\theta,\theta')\in\Theta\times\Theta'} L^c_{n,b_n}(t, \theta, \theta'), \qquad t \in [0, 1].$$

Remark In the following we will assume specific cases of the likelihood ℓ and in the appendix we will have ℓ modelled as a more general twice continuously differentiable function. A referee asked if this assumption on ℓ can be relaxed. Unfortunately, although it might be possible to relax differentiability on ℓ by using sharper and recent Gaussian approximation from [28], one can*not* possibly circumvent the covariance matrix estimation that also requires the smoothness from ℓ.

2.3. *Assumptions.* For our main results, we need the following assumptions on our time series models.

ASSUMPTION 2.1 (Case 1). *Assume that*

1. ζ_i *are i.i.d. with* $\mathbb{E}\zeta_i = 0$, $\mathbb{E}\zeta_i^2 = 1$ *and for some* $a > 0$, $\mathbb{E}|\zeta_i|^{(2+a)M} < \infty$. *Here,* $M = 3$. *In the special case* $\sigma(x,\theta)^2 \equiv \beta_0$, *one can choose* $M = 2$.
2. *For all* $t \in [0, 1]$, *the sets*

$$\{m_1(\tilde{X}_0(t)), \ldots, m_k(\tilde{X}_0(t))\}, \qquad \{\nu_0(\tilde{X}_0(t)), \ldots, \nu_l(\tilde{X}_0(t))\}$$

   are (separately) linearly independent in $\mathcal{L}_2$.
3. *There exist* $(\kappa_{ij}) \in \mathbb{R}^{k\times p}_{\geq 0}$, $(\rho_{ij}) \in \mathbb{R}^{(l+1)\times p}_{\geq 0}$ *such that for all* i:

$$(2.4) \qquad \sup_{x\neq x'} \frac{|m_i(x) - m_i(x')|}{|x - x'|_{\kappa_{i\cdot},1}} \leq 1, \qquad \sup_{x\neq x'} \frac{|\sqrt{\nu_i(x)} - \sqrt{\nu_i(x')}|}{|x - x'|_{\rho_{i\cdot},1}} \leq 1.$$

   Let $\nu_{min} > 0$ *be some constant such that for all* $x \in \mathbb{R}$, $\nu_0(x) \geq \nu_{min}$. *With some* $\beta_{min} > 0$, *choose* $\tilde{\Theta} \subset \mathbb{R}^k \times \mathbb{R}^{l+1}_{\geq \beta_{min}}$ *such that*

$$(2.5) \qquad \sum_{j=1}^{p} \left(\sup_{\theta\in\tilde{\Theta}} \sum_{i=1}^{k} |\alpha_i|\kappa_{ij} + \|\zeta_0\|_{2M} \cdot \sup_{\theta\in\tilde{\Theta}} \sum_{i=0}^{l} \sqrt{\beta_i}\rho_{ij} \right) < 1.$$

4. $\Theta \subset \tilde{\Theta}$ *is compact and for all* $t \in [0, 1]$, $\theta(t)$ *lies in the interior of* Θ. *Each component of* $\theta(\cdot)$ *is in* $C^4([0, 1])$.

5. m_i, ν_i are differentiable such that for all $j = 1, \ldots, p$ and all i,

$$\sup_{x \neq x'} \frac{|\partial_{x_j} m_i(x) - \partial_{x_j} m_i(x')|}{|x - x'|_1} < \infty, \qquad \sup_{x \neq x'} \frac{|\partial_{x_j} \nu_i(x) - \partial_{x_j} \nu_i(x')|}{|x - x'|_1} < \infty,$$

In the tvAR model (cf. [43], Example 4.1), it holds that $p = k$, $m_1(x) = x_1, \ldots, m_k(x) = x_k$, $l = 0$, $\nu_0(x) = 1$, leading to the rather strong condition $\sup_{\theta \in \Theta} \sum_{i=1}^{k} |\alpha_i| < 1$. As seen in the proof of Proposition 9.1 in the appendi, the condition (2.5) however is only needed to guarantee the existence of the process and corresponding moments. By using techniques which are more specific to the model, one can obtain much less strict assumptions such as Θ being a compact subset of

$$\{\theta = (\alpha_1, \ldots, \alpha_k, \beta_0) \in \mathbb{R}^k \times (0, \infty) : \alpha(z) = 1 + \sum_{i=1}^{k} \alpha_i z^i \text{ has only zeros outside the unit circle}\},$$

cf. [43], Example 4.1. In the tvARCH case, the above Assumption 2.1 asks for $\mathbb{E}|\zeta_1|^{6+a} < \infty$ with some $a > 0$.

In the following, we consider Case 2, the tvGARCH model. In this specific model, the moment conditions can be relaxed to $\mathbb{E}|\zeta_1|^{4+a} < \infty$. The tvGARCH model was for instance studied in the stationary case in Francq and Zakoïan [19]. More recently, pointwise asymptotic results were obtained in Rohan and Ramanathan [45]. For a matrix A, we define $\|A\|_q := (\|A_{ij}\|_q)_{ij}$ as a component-wise application of $\|\cdot\|_q$. For matrices A, B, let $A \otimes B$ denote the Kronecker product and

$$(2.6) \qquad A^{\otimes k} = A \otimes \ldots \otimes A$$

denote the k-fold Kronecker product.

ASSUMPTION 2.2 (Case 2). Let $f(\theta) = (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_l)^{\mathsf{T}}$ and let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)^{\mathsf{T}}$ be the unit column vector with jth element being 1, $1 \leq j \leq l + m$. Define $M_i(\theta) = (f(\theta)\zeta_i^2, e_1, \ldots, e_{m-1}, f(\theta), e_{m+1}, \ldots, e_{m+l-1})^{\mathsf{T}}$. Let $\alpha_{min} > 0$, and with $\rho(\cdot)$ denoting the spectral norm,

$$\tilde{\Theta} = \{\theta \in \mathbb{R}_{\geq 0}^{m+l+1} : \alpha_0 \geq \alpha_{min}, \rho(\mathbb{E}[M_0(\theta)^{\otimes 2}]) < 1\}.$$

Suppose that

(i) $\Theta \subset \text{int}(\tilde{\Theta})$ is compact and for all $t \in [0, 1]$, $\theta(t)$ lies in the interior of Θ. Each component of $\theta(\cdot)$ is in $C^4[0, 1]$,

(ii) ζ_i are i.i.d. with $\mathbb{E}\zeta_i = 0$, $\mathbb{E}\zeta_i^2 = 1$ and $\mathbb{E}|\zeta_i|^{4+a} < \infty$ with some $a > 0$.

In the important GARCH(1,1) case, the parameter space condition translates to $\alpha_0 \geq \alpha_{min}$, $\alpha_1, \beta_1 > 0$ and

$$(2.7) \qquad \beta_1^2 + 2\|\zeta_0\|_2^2 \alpha_1 \beta_1 + \alpha_1^2 \|\zeta_0\|_4^4 < 1$$

If $\zeta_0 \sim N(0,1)$, it holds that $\|\zeta_0\|_4^2 = \sqrt{3} \approx 1.73$. Bollerslev [5] proved that stationary GARCH processes have 4th moments under the exact same condition 2.7. In Section 4, Remark 4.4.1 therein, we talk about the applicability of (2.7).

3. Main results. We discuss the theoretical confidence band result in this section. As a primer it is possible to have some consistency and pointwise asymptotic results which for the sake of clear exposition have been transferred to Appendix. We directly start with a weak Bahadur representation which plays a key role for introducing simultaneity. For $l \geq 0$, define

$$\mu_{K,l} := \int K(x) x^l dx, \qquad \sigma_{K,l}^2 := \int K(x)^2 x^l dx.$$

We now have to define some quantities $V(t), I(t), \Lambda(t)$ which are needed to provide the theoretical results. They correspond to the so-called (miss-specified) Fisher information matrices which occur naturally as variance of the M-estimators. These quantities need not to be known in practice because they are estimated. They depend on the so-called *stationary approximation* $\tilde{Y}_i(t)$ of the considered time-varying process Y_i. In case 1 and case 2, this is given as follows: For $t \in [0,1]$,

- $\tilde{Y}_i(t)$ is the solution of

$$\tilde{Y}_i(t) = \mu(\tilde{Y}_{i-1}(t), ..., \tilde{Y}_{i-p}(t), \theta(t)) + \sigma(\tilde{Y}_{i-1}(t), ..., \tilde{Y}_{i-p}(t), \theta(t)), \quad i \in \mathbb{Z},$$

- $\tilde{Y}_i(t)$ is the solution of

$$
\begin{aligned}
\tilde{Y}_i(t) &= \tilde{\sigma}_i(t)^2 \zeta_i^2, \\
\tilde{\sigma}_i(t)^2 &= \alpha_0(t) + \sum_{j=1}^{m} \alpha_j(t) \tilde{Y}_{i-j}(t) + \sum_{j=1}^{l} \beta_j(t) \tilde{\sigma}_{i-j}(t)^2, \quad i \in \mathbb{Z}.
\end{aligned}
$$

For $t \in [0,1]$, let $\tilde{Z}_j(t) := (\tilde{Y}_j(t), \tilde{Y}_{j-1}(t), ...)$ denote the infinite vector containing the stationary approximations. We now define

$$(3.1) \qquad V(t) = \mathbb{E}\nabla_\theta^2 \ell(\tilde{Z}_0(t), \theta(t)),$$

$$(3.2) \qquad I(t) = \mathbb{E}[\nabla_\theta \ell(\tilde{Z}_0(t), \theta(t)) \cdot \nabla_\theta \ell(\tilde{Z}_0(t), \theta(t))^\mathsf{T}],$$

$$(3.3) \qquad \Lambda(t) = \sum_{j \in \mathbb{Z}} \mathbb{E}[\nabla_\theta \ell(\tilde{Z}_0(t), \theta(t)) \cdot \nabla_\theta \ell(\tilde{Z}_j(t), \theta(t))^\mathsf{T}].$$

In our theoretical models, these quantities can be related to each other. The following lemma (a direct implication of Propositions 9.1 and 9.2 in the appendix) summarizes these forms.

LEMMA 3.1. 　　• *Case 1: It holds that $V(t) = \Lambda(t)$.*
If additionally (i) $\mathbb{E}\zeta_0^3 = 0$, or (ii) $\mu(x,\theta) \equiv 0$ or (iii) $\sigma(x,\theta) \equiv \beta_0$ and $\mathbb{E}m(\tilde{X}_0(t)) = 0$, then

$$I(t) = \begin{pmatrix} I_k & 0 \\ 0 & (\mathbb{E}\zeta_0^4 - 1)I_{l+1}/2 \end{pmatrix} \cdot V(t),$$

where I_d denotes the d-dimensional identity matrix.
• *Case 2: It holds that $\Lambda(t) = I(t) = ((\mathbb{E}\zeta_0^4 - 1)/2)V(t)$.*

3.1. *A weak Bahadur representation for $\hat{\theta}_{b_n}, \widehat{\theta}'_{b_n}$.* In the following, we obtain a weak Bahadur representation of $\hat{\theta}_{b_n}$ and $\widehat{\theta}'_{b_n}$ which will be used to construct simultaneous confidence bands. The first part of Theorem 3.2 shows that $\hat{\theta}_{b_n}(t) - \theta(t)$ can be approximated by the expression $V(t)^{-1}\nabla_\theta L^c_{n,b_n}(t, \theta(t), \theta'(t))$ as expected due to a standard Taylor argument. The second part of Theorem 3.2 deals with approximating this term by a weighted sum of t-free terms, namely

$$(nb_n)^{-1}\sum_{i=1}^n K_{b_n}(i/n - t)h_i, \qquad h_i := \nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n)),$$

which is necessary to apply some earlier results from Zhou and Wu [59]. Similar results are obtained for $\widehat{\theta}'_{b_n}$ in Theorem 8.2(ii) in the appendix section 8. Let $\mathcal{T}_n := [b_n, 1 - b_n]$. For some vector or matrix x, let $|x| := |x|_2$ denote its Euclidean or Frobenius norm, respectively.

THEOREM 3.2 (Weak Bahadur representation of $\hat{\theta}_{b_n}, \widehat{\theta}'_{b_n}$). *Let $\beta_n = (nb_n)^{-1/2}b_n^{-1/2}\log(n)^{1/2}$ and put*

$$\tau_n^{(2)} = (\beta_n + b_n)((nb_n)^{-1/2}\log(n) + b_n^{1+j}).$$

Let Assumption 2.1 or 2.2 hold. Then it holds that

(3.4) $\qquad \sup_{t\in\mathcal{T}_n}\left|\mu_{K,2}V(t)\cdot b_n\{\widehat{\theta}'_{b_n}(t) - \theta'(t)\} - b_n^{-1}\nabla_{\theta'}L^c_{n,b_n}(t, \theta(t), \theta'(t))\right| = O_{\mathbb{P}}(\tau_n^{(2)}),$

(3.5) $\qquad \sup_{t\in\mathcal{T}_n}\left|b_n^{-1}\nabla_{\theta'}L^c_{n,b_n}(t, \theta(t), \theta'(t)) - b_n^3\frac{\mu_{K,4}}{2}V(t)\,bias(t)\right.$

$$\left. -(nb_n)^{-1}\sum_{i=1}^n K_{b_n}(i/n - t)\frac{(i/n - t)}{b_n}h_i\right| = O_{\mathbb{P}}(\beta_n b_n^2 + b_n^4 + (nb_n)^{-1}).$$

3.2. *Simultaneous confidence bands for $\hat{\theta}_{b_n}, \widehat{\theta}'_{b_n}$.* Based on the weak Bahadur result, we use results from Wu and Zhou [53] to obtain a Gaussian analogue of

$$\frac{1}{nb_n}\sum_{i=1}^n K_{b_n}(t - i/n)C^\mathsf{T}V(t)^{-1}\nabla_\theta\ell(\tilde{Z}_i(i/n), \theta(i/n)) =: \frac{1}{nb_n}\sum_{i=1}^n K_{b_n}(t - i/n)\tilde{h}_i(i/n)$$

for some $C \in \mathbb{R}^{s \times k}$. For a positive semidefinite matrix A with eigendecomposition $A = QDQ^{\mathsf{T}}$, where Q is orthonormal and D is a diagonal matrix, define $A^{1/2} = QD^{1/2}Q^{\mathsf{T}}$, where $D^{1/2}$ is the elementwise root of D.

We are now able to prove the following asymptotic statement for simultaneous confidence bands for $\theta(\cdot)$:

THEOREM 3.3 (Simultaneous confidence bands for $\theta(\cdot)$ and $\theta'(\cdot)$). *Let C be a fixed $k \times s$ matrix with rank $s \leq k$. Define $\hat{\theta}_{b_n,C}(t) := C^{\mathsf{T}} \hat{\theta}_{b_n}(t)$, $\hat{\theta}'_{b_n,C}(t) := C^{\mathsf{T}} \widehat{\theta}'_{b_n}(t)$ and $\theta_C(t) := C^{\mathsf{T}} \theta(t)$, $A_C(t) := V(t)^{-1} C$, $\Sigma_C^2(t) := A_C^{\mathsf{T}}(t) \Lambda(t) A_C(t)$.*

Let Assumption 2.1 or 2.2 be fulfilled. Assume that, for some $\alpha_{exp} < \frac{1}{2}$,

$$\log(n)\big(b_n n^{\alpha_{exp}}\big)^{-1} \to 0, \qquad nb_n^9 \log(n) \to 0.$$

Then with $\hat{K}(x) = K(x)x$,

$$\lim_{n \to \infty} \mathbb{P}\Big(\frac{\sqrt{nb_n^3}\mu_{K,2}}{\sigma_{K,2}} \sup_{t \in \mathcal{T}_n} \Big| \Sigma_C^{-1}(t)\Big\{ \hat{\theta}'_{b_n,C}(t) - \theta'_C(t) - b_n^2 \frac{\mu_{\hat{K},4}}{2\mu_{\hat{K},2}} C^{\mathsf{T}} bias(t) \Big\} \Big|$$

(3.6)
$$-B_{\hat{K}}(m^*) \leq \frac{u}{\sqrt{2\log(m^*)}} \Big) = \exp(-2\exp(-u)),$$

where in both cases $\mathcal{T}_n = [b_n, 1 - b_n]$, $m^* = 1/b_n$ and

(3.7) $$B_K(m^*) = \sqrt{2\log(m^*)} + \frac{\log(C_K) + (s/2 - 1/2)\log(\log(m^*)) - \log(2)}{\sqrt{2\log(m^*)}},$$

with

$$C_K = \frac{\Big\{ \int_{-1}^{1} |K'(u)|^2 du / \sigma_{K,0}^2 \pi \Big\}^{1/2}}{\Gamma(s/2)}.$$

Remark The conditions on b_n are fulfilled for bandwidths $b_n = n^{-\alpha}$, where $\alpha \in (0,1)$ satisfies

$$1/9 < \alpha < \alpha_{exp}.$$

The bandwidths $b_n = cn^{-1/5}$ are covered in all cases.

Remark The above theorem is a special case of Theorem 8.18 in the appendix that can cover a wide range of models.

Note that for practical use of the SCB in (3.6), one needs to estimate the bias term, choose a proper bandwidth b_n and estimate $\Sigma_C(t)$. Furthermore, the theoretical SCB only has slow logarithmic convergence, thus one requires huge n to achieve the desired coverage probability. To tackle these type of problems, we discuss practical issues in the next Section 4.

4. Implementational issues. In this section, we discuss some issues which arise by implementing the procedure from Theorem 3.3. We focus on estimation of $\hat{\theta}_{b_n}$ and optimization of the corresponding SCBs; the results for $\widehat{\theta}'_{b_n}$ can be obtained accordingly.

4.1. *Bias correction.* There are several possible ways to eliminate the bias term in Theorem 3.3. A natural way is to estimate $\theta''(t)$ by using a local quadratic estimation routine with some bandwidth $b'_n \geq b_n$. However the estimation of $\theta''(t)$ may be unstable due to the convergence condition $nb_n^5 \to \infty$ which may be hard to realize together with $nb_n^9 \log(n) \to 0$ from Theorem 3.3 in practice. Here instead we propose a bias correction via the a jack-knife method inspired from [23]. We define

$$(4.1) \qquad \tilde{\theta}_{b_n}(t) := 2\hat{\theta}_{b_n/\sqrt{2}}(t) - \hat{\theta}_{b_n}(t).$$

Since the weak Bahadur representation from Theorem 3.2 holds both for $\hat{\theta}_{b_n/\sqrt{2}}$ and $\hat{\theta}_{b_n}(t)$, we obtain

$$\sup_{t \in \mathcal{T}_n} \left| V(t) \cdot \{\tilde{\theta}_{b_n}(t) - \theta(t)\} - (nb_n)^{-1} \sum_{i=1}^{n} \tilde{K}_{b_n}(i/n - t)h_i \right| = O_{\mathbb{P}}(\tau_n^{(2)} + \beta_n b_n^2 + b_n^3 + (nb_n)^{-1}),$$

where $\tilde{K}(x) := 2\sqrt{2}K(\sqrt{2}x) - K(x)$. Note that the bias term of order b_n^2 is eliminated by construction. This shows that Theorem 3.3 still holds true for $\tilde{\theta}_{b_n}(\cdot)$ with kernel K replaced by the fourth-order kernel $\tilde{K}$ and with no bias term of order b_n^2.

4.2. *Estimation of the covariance matrix $\Sigma_C(t)$.* In this subsection, we discuss the estimation of $\Sigma_C^2(t)$ (namely, $V(t)$ and $\Lambda(t)$) since this term is generally unknown but arises in the SCB in Theorem 3.3. From Lemma 3.1 it can be seen that in many cases, it holds that $\Lambda(t) = I(t)$ due to the fact that the $\nabla_\theta \ell(\tilde{Z}_i(t), \theta(t))$, $i \in \mathbb{Z}$ are uncorrelated. In the case that the objective function ℓ coincides with the true log conditional likelihood, one has even $V(t) = I(t)$. Even in the misspecified case it may often hold that $V(t) = c_0 \cdot I(t)$ with some constant $c_0 > 0$ only dependent on properties of the i.i.d. innovations ζ_0 which can be calculated by further assumptions on ζ_0.

Therefore, it may often hold that $\Sigma_C^2(t) = C^\mathsf{T} V(t)^{-1} \Lambda(t) V(t)^{-1} C$ obeys one of the two equalities

$$(4.2) \qquad \Sigma_C^2(t) \;=\; C^\mathsf{T} V(t)^{-1} I(t) V(t)^{-1} C, \qquad \text{or}$$
$$(4.3) \qquad \Sigma_C^2(t) \;=\; C^\mathsf{T} I(t)^{-1} C / c_0 \quad \text{with some known constant } c_0.$$

We therefore focus on estimation of $V(t)$ and $I(t)$. We propose the (boundary-corrected)

estimators

$$(4.4)\ \hat{V}_{b_n}(t) \quad := \quad (nb_n\hat{\mu}_{K,0,b_n}(t))^{-1}\sum_{i=1}^{n}K_{b_n}(i/n-t)\nabla_{\theta}^2\ell(Z_i^c,\hat{\theta}_{b_n}(t)+(i/n-t)\widehat{\theta'}_{b_n}(t)),$$

$$(4.5)\ \hat{I}_{b_n}(t) \quad := \quad (nb_n\hat{\mu}_{K,0,b_n}(t))^{-1}\sum_{i=1}^{n}K_{b_n}(i/n-t)\nabla_{\theta}\ell(Z_i^c,\hat{\theta}_{b_n}(t)+(i/n-t)\widehat{\theta'}_{b_n}(t))$$
$$\times\nabla_{\theta}\ell(Z_i^c,\hat{\theta}_{b_n}(t)+(i/n-t)\widehat{\theta'}_{b_n}(t))^{\mathsf{T}},$$

where $\hat{\mu}_{K,0,b_n}(t) := \int_{-t/b_n}^{(1-t)/b_n}K(x)dx$. The convergence of these estimators is given in the next Proposition. Note that the following Proposition also holds if $\widehat{\theta'}_{b_n}$ in (4.4) and (4.5) is replaced by 0.

PROPOSITION 4.1. *Let Assumption 2.1 or 2.2 hold. Let $(\beta_n + b_n)\log(n)^2 \to 0$. Then*

(i) $\sup_{t\in(0,1)}|\hat{V}_{b_n}(t) - V(t)| = O_{\mathbb{P}}((\log n)^{-1})$.

(ii) *If $r > 4$, then* $\sup_{t\in(0,1)}|\hat{I}_{b_n}(t) - I(t)| = O_{\mathbb{P}}((\log n)^{-1})$.

This shows uniform consistency of $\hat{V}_{b_n}(\cdot)$, $\hat{I}_{b_n}(\cdot)$ if $(\beta_n + b_n)\log(n)^2 \to 0$. Note that in (ii), we need more moments to discuss $\nabla_{\theta}\ell \cdot \nabla_{\theta}\ell^{\mathsf{T}} \in \mathcal{H}(2M_y, 2M_x, \chi, \bar{\bar{C}})$ $(\bar{\bar{C}} > 0)$. In many special cases, this may be relaxed.

In either case (4.2) or (4.3), we define $\hat{\Sigma}_C(t)$ by replacing $V(t), I(t)$ by the corresponding estimators $\hat{V}_{b_n}(t)$, $\hat{I}_{b_n}(t)$.

4.3. *Bandwidth selection.* Based on the asymptotic variance given in the asymptotic normality result in Theorem 8.1 in the appendix, the MSE global optimal bandwidth choice reads

$$(4.6) \qquad \hat{b}_n = n^{-1/5}\cdot\Big(\frac{\sigma_{K,0}^2\int_0^1\mathrm{tr}(V(t)^{-1}I(t)V(t)^{-1})dt}{\mu_{K,2}^2\int_0^1|\theta''(t)|^2dt}\Big)^{1/5}.$$

We therefore adapt a model-based cross validation method from Richter and Dahlhaus [43], which was shown to work even if the underlying parameter curve is only Hölder continuous and $\nabla_{\theta}\ell(\tilde{Z}_i(t),\theta(t))$ is uncorrelated. Here, we reformulate this selection procedure for the local linear setting. For $j = 1,\ldots,n$, define the leave-one-out local linear likelihood

$$(4.7) \qquad L_{n,b_n,-j}^c(t,\theta,\theta') := (nb_n)^{-1}\sum_{i=1,i\neq j}^{n}K_{b_n}(i/n-t)\ell(Z_i^c,\theta+(i/n-t)\theta')$$

and the corresponding leave-one-out estimator

$$(\hat{\theta}_{b_n,-j}(t),\hat{\theta}'_{b_n,-j}(t)) = \operatorname*{argmin}_{\theta\in\Theta,\theta'\in\Theta'}L_{n,b_n,-j}^c(t,\theta,\theta').$$

The bandwidth $\hat{b}_n^{CV}$ is chosen via minimizing

$$(4.8) \qquad CV(b_n) := n^{-1} \sum_{i=1}^{n} \ell(Z_i^c, \hat{\theta}_{b_n,-i}(i/n)) w(i/n),$$

where $w(\cdot)$ is some weight function to exclude boundary effects. A possible choice is $w(\cdot) := \mathbf{1}_{[\gamma_0, 1-\gamma_0]}$ with some fixed $\gamma_0 > 0$. Note that it is important to use the modified local linear approach due to the different bias terms. In Richter and Dahlhaus [43], it was shown that the local constant version of this procedure selects asymptotically optimal bandwidths and works even if a model misspecification is present, i.e. if the function ℓ leads to estimators $\hat{\theta}_{b_n}$ which are not consistent. This motivates that a similar behavior should hold for the local constant version.

4.4. *Bootstrap method.* The SCB for $\theta_C(t)$ obtained in Theorem 3.3 provides a slow logarithmic rate of convergence to the Gumbel distribution. Thus, for even moderately large values of sample size n, it is practically infeasible to use such a theoretical SCB as the coverage will possibly be lower than the specified nominal level. First we show an empirical coverage comparison of how far the theoretical confidence intervals lag behind in achieving their nominal coverage. We use the same simulation setting (cf. section 6.1) for the tvGARCH case:

$$X_i = \sigma_i \zeta_i, \sigma_i^2 = \alpha_0(i/n) + \alpha_1(i/n) X_{i-1}^2 + \beta_1(i/n) \sigma_{i-1}^2,$$

where $\alpha_0(t) = 1.0 + 0.2 \sin(2\pi t)$, $\alpha_1(t) = 0.45 + 0.1 \sin(\pi t)$ and $\beta_1(t) = 0.1 + 0.1 \sin(\pi t)$, ζ_i is i.i.d. standard normal distributed. For estimation, we choose $K(x) = \frac{3}{4}(1 - x^2) \mathbf{1}_{[-1,1]}(x)$ to be the Epanechnikov kernel, $n = 2000, 5000$ for several different b_n. From Table 1 one can see, the simultaneous coverage is never even positive for the throetical ones. The individual coverages are very low for small bandwidth and with higher bandwidth they overcompensate. Also, the performance of $n = 5000$ is slightly better hinting at logarithmic convergence rate.

We circumvent this convergence issue in this subsection by proposing a wild bootstrap algorithm. Recall the jackknife-based bias corrected estimator of $\tilde{\theta}_{b_n}$ from (4.1). Let $\tilde{\theta}_C(t) = C^T \tilde{\theta}_{b_n}(t)$. We have the following proposition as the key idea behind the bootstrap method.

PROPOSITION 4.2. *Suppose that Assumption 2.1 or Assumption 2.2 holds. Furthermore, assume that $b_n = O(n^{-\kappa})$ with $1/9 < \kappa < \frac{1}{2}$. Then on a richer probability space, there are i.i.d. $V_1, V_2, \ldots, \sim N(0, Id_s)$ such that*

$$(4.9) \qquad \sup_{t \in \mathcal{T}_n} |\hat{\theta}_{b_n,C}(t) - \theta_C(t) - \Sigma_C(t) Q_{b_n}^{(0)}(t)| = O_{\mathbb{P}}\big(\frac{n^{-\nu}}{\sqrt{nb_n} \log(n)^{1/2}}\big),$$

TABLE 1
Coverage probabilities of the SCB in (b) for $n = 500, 1000, 2000, 5000$, $b_n^{opt} = 0.49, 0.41$ respectively

n	b_n	$\alpha = 90\%$				$\alpha = 95\%$			
		α_0	α_1	β_1	$(\alpha_0, \alpha_1, \beta_1)^{\mathsf{T}}$	α_0	α_1	β_1	$(\alpha_0, \alpha_1, \beta_1)^{\mathsf{T}}$
2000	0.3 Gumbel	0.778	0.405	0.638	0	0.629	0.215	0.42	0
	0.3 Bootstrap	0.944	0.822	0.924	0.869	0.967	0.89	0.961	0.92
	0.35 Gumbel	0.954	0.891	0.94	0	0.871	0.714	0.818	0
	0.35 Bootstrap	0.941	0.843	0.923	0.864	0.967	0.908	0.945	0.913
5000	0.25 Gumbel	0.6	0.248	0.481	0	0.481	0.126	0.298	0
	0.25 Bootstrap	0.955	0.855	0.936	0.886	0.974	0.903	0.959	0.929
	0.30 Gumbel	0.756	0.461	0.642	0	0.643	0.261	0.432	0
	0.30 Bootstrap	0.941	0.889	0.939	0.904	0.974	0.931	0.967	0.949
	0.35 Gumbel	0.96	0.917	0.957	0	0.878	0.701	0.84	0
	0.35 Bootstrap	0.949	0.903	0.941	0.903	0.972	0.95	0.975	0.946

where $\nu = \min\{\frac{1}{4} - \kappa/2, 7\kappa/2 - 1/2, \kappa/2\} > 0$ and

$$Q_{b_n}^{(0)}(t) = \frac{1}{nb_n} \sum_{i=1}^{n} V_i K_{b_n}(i/n - t).$$

The proof of Proposition 4.2 is immediate from the approximation rates (8.74), (8.75), (8.80) and (8.82) which, ignoring the $\log(n)$ terms, are of the form $c_n \cdot (nb_n)^{-1/2} \log(n)^{-1/2}$ with

$$c_n \in \{\big(b_n n^{(2\gamma + \varsigma\gamma - \varsigma)/(\varsigma + 4\gamma + 2\gamma\varsigma)}\big)^{-1/2}, b_n^{1/2}, b_n, (nb_n^7)^{1/2}, (nb_n^2)^{-1/2}\},$$

where $\gamma > 1$ is arbitrarily large and $\varsigma > 0$.

One can interpret (4.9) in the sense that $\Sigma_C(t) Q_{b_n}^{(0)}(t)$ approximates the stochastic variation in $\hat{\theta}_{b_n, C}(t) - \theta_C(t)$ uniformly over $t \in \mathcal{T}_n$ and thus it can be used as margin of errors to construct confidence bands, provided one can consistently estimate $\Sigma_C(t)$.

4.4.1. *Boundary considerations.* The results shown above only hold for $t \in \mathcal{T}_n$. For inference of some time series models like ARCH or GARCH, large bandwidths are needed to get sufficiently smooth and stable estimators even for a large number of observations. It seems hard to generalize the SCB result Theorem 3.3 to the whole interval $t \in (0, 1)$. However it is possible to generalize the bootstrap procedure which may be more important in practice:

PROPOSITION 4.3. *Suppose that the conditions on κ, ν of Proposition 4.2 hold. Then on a richer probability space, there exist i.i.d. $V_1, V_2, \ldots, \sim N(0, Id_s)$ such that*

$$\sup_{t \in (0,1)} |N_{b_n}^{(0)}(t) \cdot \{\hat{\theta}_{b_n, C}(t) - \theta_C(t)\} + b_n^2 N_{b_n}^{(1)}(t) \theta_C''(t) - \Sigma_C(t) W_{b_n}(t)| = O_{\mathbb{P}}\big(\frac{n^{-\nu}}{\sqrt{nb_n} \log(n)^{1/2}}\big),$$

where

$$(4.10) \qquad W_{b_n}(t) = Q_{b_n}^{(0)}(t) - \frac{\hat{\mu}_{K,1,b_n}(t)}{\hat{\mu}_{K,2,b_n}(t)} \cdot Q_{b_n}^{(1)}(t)$$

and $N_{b_n}^{(j)}(t) := \frac{\hat{\mu}_{K,j,b_n}(t)\hat{\mu}_{K,j+2,b_n}(t) - \hat{\mu}_{K,j+1,b_n}(t)^2}{\hat{\mu}_{K,2,b_n}(t)}$, $\hat{\mu}_{K,j,b_n}(t) := \int_{-t/b_n}^{(1-t)/b_n} K(x)x^j dx$,

$$Q_{b_n}^{(j)}(t) = \frac{1}{nb_n}\sum_{i=1}^{n} V_i K_{b_n}(i/n - t)\big[(i/n - t)b_n^{-1}\big]^j, \qquad (j = 0, 1).$$

Note that the additional term in (4.10) reduces to $Q_{b_n}^{(0)}(t)$ for $t \in \mathcal{T}_n$.

To eliminate the bias inside $t \in \mathcal{T}_n$ it is still recommended to use the jack-knife estimator $\tilde{\theta}_C(t)$. From Proposition 4.3 we obtain

$$\sup_{t \in (0,1)} \big| N_{b_n}^{(0)}(t)N_{b_n/\sqrt{2}}^{(0)}(t)\{\tilde{\theta}_C(t) - \theta(t)\} + b_n^2\{N_{b_n/\sqrt{2}}^{(1)}(t)N_{b_n}^{(0)}(t) - N_{b_n}^{(1)}(t)N_{b_n/\sqrt{2}}^{(0)}(t)\}\theta_C''(t)$$

$$(4.11) \qquad\qquad -\Sigma_C(t)W_{b_n}^{(debias)}(t)\big| = O_{\mathbb{P}}\Big(\frac{n^{-\nu}}{\sqrt{nb_n}\log(n)^{1/2}}\Big),$$

where
(4.12)
$$W_{b_n}^{(debias)}(t) = 2N_{b_n}^{(0)}(t)\cdot\Big[Q_{b_n/\sqrt{2}}^{(0)}(t) - \frac{\hat{\mu}_{K,1,b_n/\sqrt{2}}(t)}{\hat{\mu}_{K,2,b_n/\sqrt{2}}(t)}Q_{b_n/\sqrt{2}}^{(1)}(t)\Big] - N_{b_n/\sqrt{2}}^{(0)}(t)\cdot\Big[Q_{b_n}^{(0)}(t) - \frac{\hat{\mu}_{K,1,b_n}(t)}{\hat{\mu}_{K,2,b_n}(t)}Q_{b_n}^{(1)}(t)\Big].$$

The additional factor $N_{b_n}^{(0)}(t)N_{b_n/\sqrt{2}}^{(0)}(t)$ in (4.11) serves as an indicator how near t is to the boundary. For $t \in \mathcal{T}_n$, this factor is 1 while for $t \in (0,1)\backslash\mathcal{T}_n$, $N_{b_n}^{(0)}(t)N_{b_n/\sqrt{2}}^{(0)}(t)$ may be very small, inducing large diameters of the band near the boundary. Note that the bias correction of the jack-knife estimator $\tilde{\theta}_C(t)$ may be useless in $t \in (0,1)\backslash\mathcal{T}_n$ since $N_{b_n/\sqrt{2}}^{(1)}(t)N_{b_n}^{(0)}(t) \neq N_{b_n}^{(1)}(t)N_{b_n/\sqrt{2}}^{(0)}(t)$. However it is necessary from a theoretical point of view to use the same estimator for the whole region $(0,1)$ to get a uniform band based on the approximation (4.11).

In practice, the result (4.11) can be used as follows: We can create a large number of i.i.d. copies $W_{b_n}^{(boot,debias)}(t)$ of $W_{b_n}^{(debias)}(t)$ by creating i.i.d. copies

$$(4.13)$$

$$Q_{b_n}^{(0),boot}(t) = \frac{1}{nb_n}\sum_{i=1}^{n} V_i^* K_{b_n}(i/n - t), \qquad Q_{b_n}^{(1),boot} \frac{1}{nb_n}\sum_{i=1}^{n} V_i^* K_{b_n}(i/n - t)\cdot(i/n - t)b_n^{-1}$$

where $V_1^*, V_2^*, \ldots$, are i.i.d. $N(0, I_{s\times s})$-distributed random variables, and computing $W_{b_n}^{(boot,debias)}(t)$ according to (4.12). Quantiles of $W_{b_n}^{(debias)}(t)$ then can be determined by using the corresponding empirical quantile of the copies $W_{b_n}^{(boot,debias)}(t)$. Then one can use (4.11) to

construct the confidence band for $\theta_C(t)$. For convenience of the readers, we provide a summarized algorithm of the above discussion.

Algorithm for constructing SCBs of $\theta_C(t)$:

- Compute the appropriate bandwidth b_n based on the cross validation method in Subsection 4.3 and compute $\tilde{\theta}_C(t)$ based on the jackknife-based estimator from 4.1.
- For $r = 1, \ldots, N$ with some large N, generate n i.i.d. $N(0, I_{s \times s})$ random variables $V_1^*, \ldots, V_n^*$ and compute $q_r = \sup_{t \in (0,1)} |W_{b_n}^{(boot,debias)}(t)|$, where $W_{b_n}^{(boot,debias)}(t)$ is computed according to (4.12), (4.13).
- Compute $u_{1-\alpha} = q_{\lfloor (1-\alpha)N \rfloor}$, the empirical $(1-\alpha)$th quantile of $\sup_{t \in [0,1]} |W_{b_n}^{(debias)}(t)|$.
- Calculate $\hat{\Sigma}_C(t) = \{C^T \hat{V}(t)^{-1} \hat{\Lambda}(t) \hat{V}(t)^{-1} C\}^{1/2}$ with the estimators proposed in Subsection 4.2. As mentioned there, $V(t)^{-1} \Lambda(t) V(t)^{-1}$ can often be simplified.
- The SCB for $\theta_C(t)$ is $\tilde{\theta}_{C,b_n}(t) + \hat{\Sigma}_C(t) u_{1-\alpha} \mathcal{B}_s$, where $\mathcal{B}_s = \{x \in \mathbb{R}^s : |x| \leq 1\}$ is the unit ball in $\mathbb{R}^s$.

Remark (Discussion of the tvGARCH parameter restriction) A very valid question was asked by a reviewer about the applicability of the assumption (2.7). We would like to point out that this assumption is necessary under the fourth moment assumption of the GARCH process. Investigating the proof of Proposition 9.2 very minutely, it seems that it might be possible to relax existence of $4 + a$ moments for the GARCH process to only $2 + a$ moments which could potentially improve the condition (2.7) to

$$\alpha_1(\cdot) + \beta_1(\cdot) < 1.$$

However, the entire bias expansion arguments in the proof of Theorem 8.18 would change based on this relaxed moment assumption and it would require a different notion of local-stationarity that allows more approximating terms. To keep the general theme of the paper, we decided against proving a separate result for just GARCH(1,1). Moreover, from a practical point of view when we estimate the $\Sigma_C(t)$ we use $I_{b_n}(t)$ from section 4 which is only consistent under at least 4th moment existence of the GARCH process. We also found that, for some very popular stock market datasets (one such example is given in Section 5) one can reasonably assume that the condition (2.7) is satisfied.

5. Simulation results and applications. This section consists of some summarized simulations and some real data applications related to our theoretical results. Because of the generality of our theoretical framework, it is impossible to report simulation performance even for the most prominent examples in these different classes. Therefore we restrict ourselves to conditional heteroscedasticity (CH) models for simulations and real data applications. For the time-varying simultaneous band, to the best of our knowledge, there is no or little simulation results reported. For the tvAR, tvMA, tvARMA and tvRegressions we obtained satisfactory results but they are omitted here to keep this discussion concise.

5.1. *Simulations.* In this section, we study the finite sample coverage probabilities of our SCBs for theoretical coverage $\alpha = 0.9$ and $\alpha = 0.95$ in the following tvARCH(1) and tvGARCH(1,1) models:

(a) $X_i = \sqrt{\alpha_0(i/n) + \alpha_1(i/n)X_{i-1}^2}\,\zeta_i$, where $\alpha_0(t) = 0.8 + 0.3\cos(\pi t)$, $\alpha_1(t) = 0.45 + 0.1\cos(\pi t)$,

(b) $X_i = \sigma_i\zeta_i$, $\sigma_i^2 = \alpha_0(i/n) + \alpha_1(i/n)X_{i-1}^2 + \beta_1(i/n)\sigma_{i-1}^2$, where $\alpha_0(t) = 2.4 + 0.02\cos(\pi t)$, $\alpha_1(t) = 0.4 + 0.1\cos(\pi t)$ and $\beta_1(t) = 0.5 - 0.1\cos(\pi t)$,

where ζ_i is i.i.d. standard normal distributed. For estimation, we choose $K(x) = \frac{3}{4}(1 - x^2)\mathbf{1}_{[-1,1]}(x)$ to be the Epanechnikov kernel, $n = 500, 1000, 2000, 5000$ for several different b_n (the optimal bandwidths (4.6) are also reported for model (a) and model (b)). For each situation, $N = 2000$ replications are performed and it is checked if the obtained SCB based on (4.11) contains the true curves in $t \in (0, 1)$. In both models we have $\Lambda(t) = I(t) = V(t)$ and therefore estimate $\Sigma_C^2(t) = C^{\mathsf{T}}I(t)^{-1}C$ via replacing $I(t)$ by $\hat{I}_{b_n}(t)$ from (4.5). We obtained the results given in Tables 2 and 3. The estimation, for smaller sample sizes n, sometimes may lead to difficulties since the optimization routine (*optim* in programming language R) may not converge. We decided to discard these pathological cases for simplicity. It can be seen that the empirical coverage probabilities are reasonably close to the nominal level for bandwidths close to the optimal ones and they do not differ too much for other bandwidths as well.

TABLE 2
Coverage probabilities of the SCB in (a) for $n = 500, 1000, 2000, 5000$;

n	b_n	$\alpha = 90\%$			$\alpha = 95\%$		
		α_0	α_1	$(\alpha_0, \alpha_1)^{\mathsf{T}}$	α_0	α_1	$(\alpha_0, \alpha_1)^{\mathsf{T}}$
500	0.45	0.859	0.839	0.833	0.948	0.912	0.900
	0.5	0.869	0.864	0.842	0.937	0.914	0.895
	0.55	0.864	0.849	0.832	0.930	0.901	0.898
1000	0.4	0.873	0.846	0.845	0.937	0.906	0.900
	0.45	0.885	0.875	0.879	0.941	0.925	0.927
	0.5	0.887	0.876	0.864	0.948	0.926	0.931
	0.55	0.871	0.870	0.866	0.931	0.925	0.921
2000	0.3	0.893	0.861	0.868	0.946	0.924	0.930
	0.35	0.886	0.872	0.866	0.938	0.928	0.921
	0.4	0.891	0.878	0.874	0.937	0.926	0.933
	0.45	0.874	0.873	0.883	0.940	0.937	0.937
5000	0.25	0.885	0.883	0.882	0.941	0.931	0.936
	0.3	0.892	0.883	0.889	0.949	0.938	0.941
	0.35	0.900	0.891	0.894	0.948	0.945	0.938
	0.4	0.900	0.899	0.894	0.953	0.947	0.937
	0.45	0.878	0.880	0.881	0.934	0.937	0.930

Table 3

Coverage probabilities of the SCB in (b) for $n = 500, 1000, 2000, 5000$

n	b_n	$\alpha = 90\%$				$\alpha = 95\%$			
		α_0	α_1	β_1	$(\alpha_0, \alpha_1, \beta_1)^{\mathsf{T}}$	α_0	α_1	β_1	$(\alpha_0, \alpha_1, \beta_1)^{\mathsf{T}}$
500	0.55	0.943	0.778	0.902	0.793	0.962	0.851	0.939	0.854
	0.6	0.946	0.815	0.922	0.837	0.962	0.875	0.961	0.89
	0.65	0.947	0.828	0.92	0.841	0.966	0.884	0.956	0.899
	0.7	0.939	0.849	0.922	0.853	0.962	0.897	0.957	0.903
1000	0.5	0.943	0.857	0.936	0.864	0.974	0.913	0.966	0.914
	0.55	0.947	0.869	0.926	0.891	0.97	0.929	0.954	0.937
	0.6	0.946	0.884	0.943	0.908	0.971	0.937	0.966	0.948
	0.65	0.944	0.873	0.921	0.889	0.965	0.921	0.951	0.934
2000	0.3	0.944	0.822	0.924	0.869	0.967	0.89	0.961	0.92
	0.35	0.941	0.843	0.923	0.864	0.967	0.908	0.945	0.913
	0.40	0.958	0.846	0.92	0.894	0.972	0.91	0.96	0.943
	0.45	0.957	0.875	0.931	0.897	0.98	0.928	0.965	0.942
	0.50	0.946	0.887	0.952	0.911	0.978	0.938	0.979	0.952
5000	0.25	0.955	0.855	0.936	0.886	0.974	0.903	0.959	0.929
	0.30	0.941	0.889	0.939	0.904	0.974	0.931	0.967	0.949
	0.35	0.949	0.903	0.941	0.903	0.972	0.95	0.975	0.946
	0.40	0.954	0.882	0.94	0.92	0.968	0.94	0.969	0.959
	0.45	0.966	0.892	0.956	0.909	0.98	0.946	0.98	0.96
	0.50	0.948	0.886	0.932	0.896	0.976	0.937	0.976	0.957

5.2. *Applications.* In this section, we consider a few real-data applications of our procedure. As mentioned in Section 1, there are abundant results in the literature about time-varying regression but the results for time-varying autoregressive conditional heteroscedastic models are scarce. Thus it is important to evaluate the performance of our constructed SCBs for these type of models in both theoretical and real data scenarios. Among the popular heteroscedastic models, usually GARCH type models are most difficult to estimate due to the recursion of the variance term.

We consider two examples from the class of conditional heteroscedastic models with two types of financial datasets: one foreign exchange and one stock market daily pricing data. As Fryzlewicz, Sapatinas and Subba Rao [20] found out, ARCH models have good forecasting ability for currency exchange type data whereas for data coming from the stock market, GARCH models are preferred. Typically, these daily closing price data show unit root behavior and thus instead of using the daily price data, we model the log-return data. The log-return is defined as follows and is close to the relative return

$$Y_i = \log P_i - \log P_{i-1} = \log\left(1 + \frac{P_i - P_{i-1}}{P_{i-1}}\right) \approx \frac{P_i - P_{i-1}}{P_{i-1}},$$

where P_i is the closing price on the ith day. Because of the apparent time-varying nature of volatility these log-return data typically show, conditional heteroscedastic models are used

for analysis and forecasting.

5.2.1. *Real data application I: USD/GBP rates.* For the first application, we consider a tvARCH(p) model with $p = 1, 2$. It has the following form

$$Y_i^2 = \sigma_i^2 \zeta_i^2, \qquad \sigma_i^2 = \alpha_0(i/n) + \alpha_1(i/n)Y_{i-1}^2 + \ldots + \alpha_p(i/n)Y_{i-p}^2.$$

Many different exchange rates from 1990-1999 for USD with other currencies were analyzed in [20] using tvARCH(p) models with $p = 0, 1, 2$. The authors suggested choosing $p = 1$ for USD-GBP exchange rates. We collect the same data from www.federalreserve.gov/releases/h10/Hist/default1999.htm and fit both tvARCH(1) and tvARCH(2) models. This is a sample of size 2514 and we use cross-validated bandwidth 0.26 and 0.25 for the two models. We only show the results for the fit with tvARCH(1) here. We observed that the estimates for the parameter curves $\alpha_0(\cdot)$ and $\alpha_1(\cdot)$ for tvARCH(2) model are very similar to that from the tvARCH(1) fit and thus it indicates against including the extra $\alpha_2(\cdot)$ parameter in our model. We also provide the plots for the log-returns and ACF plot of squared sample that shows evidence of conditional heteroscedasticity.
Based on Figure 1 time-constancy for parameter curve $\alpha(\cdot)$ is rejected at 5% level of significance. For $\alpha_1(\cdot)$, the estimate generally stays below the stationary fit. Also, one can see from the plot of actual log-returns that there are large shocks from 1990 to 1993 compared to those seen in 1993-1999. This can be explained through the high (low) values shown for the estimated curve $\alpha_0(\cdot)$ for the time-period 1990-1993 (1993-1999).

5.2.2. *Real data application II: NASDAQ index data.* In the empirical analysis of log-return for stock market data, however, as Palm [40] and others suggest, lower order GARCH have been often found to account sufficiently for the conditional heteroscedasticity. Moreover, GARCH(1,1) and in a very few cases GARCH(1,2) and GARCH(2,1) models are used and higher order GARCH models are typically not necessary. Another advantage of using GARCH(1,1) over ARCH(p) models is that one need not worry about choosing a proper lag p as GARCH(1,1) can be thought as an ARCH model with $p = \infty$. In this subsection, we implement a time-varying version of GARCH(1,1) and obtain the bootstrapped SCB. A tvGARCH(1,1) model has the following form:

$$Y_i^2 = \sigma_i^2 \zeta_i^2, \qquad \sigma_i^2 = \alpha_0(i/n) + \alpha_1(i/n)Y_{i-1}^2 + \beta_1(i/n)\sigma_{i-1}^2.$$

As our second example, we choose to analyze the log returns of NASDAQ from January 2011 to December 2018. This is an extremely important index data in US stock market. We collect this one and all other stock index datasets from www.investing.com. Our cross-validated bandwidth is 0.405 for this data of size 1751. Since our simulations show excellent performance for sample sizes smaller than 2000 and the estimated parameter functions satisfy the parameter restriction $\sup_{0 \leq t \leq 1}(\hat{\beta}_1(t)^2 + 2\hat{\alpha}_1(t)\hat{\beta}_1(t) + 3\hat{\alpha}_1(t)^2) < 1$, it is reasonable to say our simultaneous confidence bands would also be valid here. As one
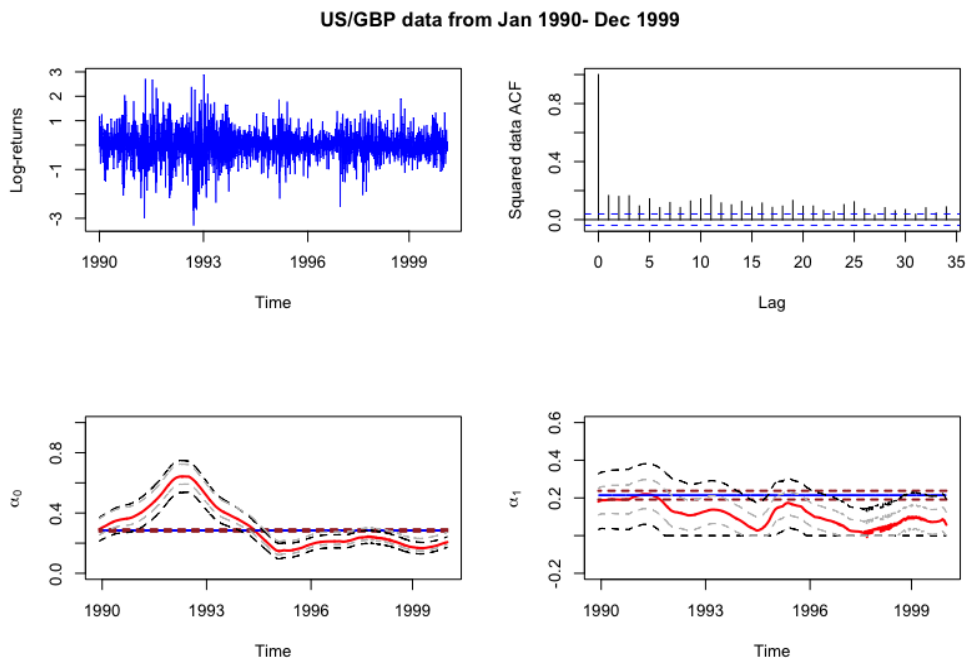
US/GBP data from Jan 1990- Dec 1999

FIG 1. *Analysis of USD/GBP data from Jan 1990 to Dec 1999. Top left: Log-returns. Top right: ACF plot. Bottom panel: Estimates of the parameters $\alpha_0(\cdot)$, $\alpha_1(\cdot)$, respectively (red) with 95% SCBs (dashed) and estimates of the parameters assuming constancy (blue). We also provide pointwise bands (grey) and time-constant estimate $\pm$ standard error band (brown). Optimal bandwidth was 0.26 and $\alpha_0(\cdot)$ is time-varying*

can see from Figure 2, the time series show significant lags in its ACF plot after squaring; indicating conditional heteroscedasticity.

One can see that the estimates for $\alpha_0(\cdot)$ is mostly above the corresponding time-constant fit. As mentioned int he caption $\alpha_1(\cdot)$ is time-varying since the SCB does not contain a horizontal line. The fit fluctuates around the time-constant fit. For $\beta_1(t)$, the time-varying fit is below the corresponding time-constant fit. Overall, since $\alpha_1(t)$ is deemed time-varying through this analysis the time-constant hypothesis can be rejected at 5% level of significance.

5.3. *Forecasting volatility.* A compelling question can be asked regarding the usefulness of time-varying models in forecasting econometric time-series compared to their time-constant analogue. Note that the main goal of this paper is not to build better forecasting models and the extension to predictive intervals from confidence intervals for conditional heteroscedastic model is not very straight-forward. Moreover, it is unclear how in-fill asymptotics discussed in this paper would extend to forecasting future trends or future time-varying function. Any asymptotic theory would need to consider the rescaling mecha-
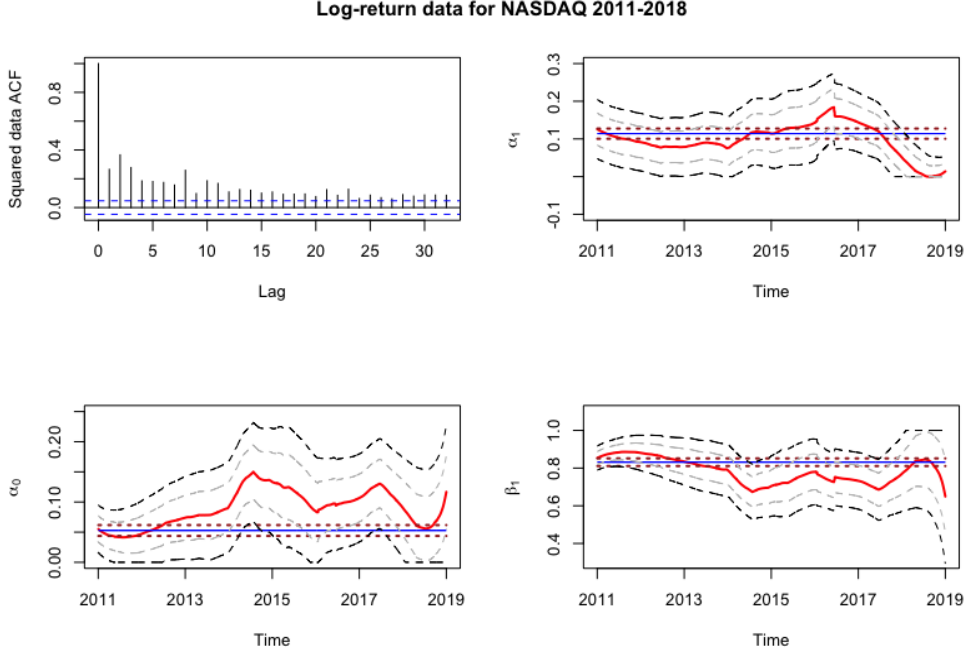
FIG 2. *Analysis of Nasdaq data from Jan 2011 to Dec 2018. Top left: ACF plot. Top right, bottom left, bottom right: Estimates of the parameters $\alpha_0(\cdot)$, $\alpha_1(\cdot)$ and $\beta_1(\cdot)$, respectively (red) with SCBs (dashed) and estimates of the parameters assuming constancy (blue). We also provide pointwise bands (grey) and time-constant estimate $\pm$ standard error band (brown). Optimal bandwidth was 0.405. A horizontal line does not pass through the SCB of $\alpha_1(\cdot)$ and thus it is time-varying.*

nism rigorously keeping in mind the data observed upto a certain point. In this subsection we show that time-varying models can indeed lead to better forecast compared to time-constant analogues.

5.3.1. *Short-range forecasts.* Following [46] and [21], we show for a wide range of econometric datasets time-varying model can provide better short range forecasts. We also allow multiple windows of forecasting and multiple start points to highlight why most of these datasets call for a time-varying fit. In the following Tables 4 and 5, we use ARCH(1) models for the forex datsets and GARCH(1,1) for the stock market indices. Our POOS (pseudo-out-of-sample) evaluation of forecasting is chalked out as follows:

Define, for a $h-$step ahead forecasting scheme, $\bar{\sigma}^2_{t,t+h} = \frac{1}{h} \sum_{i=t+1}^{t+h} \sum \sigma^2_{i|t}$ where $\sigma^2_{i|t}$ are the $(i-t)$ step ahead forecasts of time-constant or time-varying fit at time t. We compare this with 'realized' volatility $\bar{X}^2_{t,t+h} = \frac{1}{h} \sum_{i=t+1}^{t+h} X^2_i$. We then compute the aggregated

measure for a startpoint s as following

$$AMSE = \frac{1}{n-h-s} \sum_{s+1}^{n-h} (\bar{\sigma}_{t,t+h}^2 - \bar{X}_{t,t+h}^2).$$

We choose forecasting horizon h values to be 25,50,75,100,150,200 and start values to be $s = 500, 1000$. Our forecasting method is same as that is implemented in fGARCH R-package. For the time-constant fit at time t we use the data from 1 to t to predict the h-step ahead forecast. For the time-varying fit however, it is unclear what the time-varying projection will be. Following [21] we assume the last m points to be stationary for a small m and use that to predict the future forecasts. Since m is a tuning parameter we choose the m that produces the minimum $AMSE$ over the choice of $m = 100, 200, \ldots, 500$.

TABLE 4

Forecasting volatility: Short range forecast comparison of time-varying and time-constant model for startpoint at 500. TV and TC stands for time-varying and time-constant models

| | ahead=25 | | ahead=50 | | ahead=75 | | ahead=100 | | ahead=150 | | ahead=200 | |
Data	TC	TV	TC	TV	TC	TV	TC	TV	TC	TV	TC	TV
USGBP	0.101	0.07	0.085	0.057	0.078	0.053	0.075	0.052	0.07	0.048	0.067	0.046
USCHF	2.968	2.775	1.702	1.528	1.269	1.115	1.048	0.099	0.803	0.601	0.646	0.524
USCAD	0.029	0.026	0.024	0.024	0.022	0.023	0.02	0.021	0.017	0.018	0.015	0.017
EURGBP	0.083	0.074	0.057	0.048	0.044	0.036	0.039	0.032	0.034	0.029	0.028	0.025
EURUSD	0.052	0.033	0.044	0.028	0.041	0.027	0.042	0.028	0.038	0.032	0.036	0.038
SP Merval	7.722	7.094	6.632	4.679	5.317	4.376	5.763	4.514	5.372	4.529	5.086	4.620
BSE	3219	3038	2916	1901	2313	2040	2259	1321	2000	1106	1897	1038
SP500 old	5.475	5.346	8.752	7.511	4.265	4.048	3.339	2.344	2.092	1.079	1.768	0.939
SP500	0.319	0.042	0.379	0.372	0.249	0.324	0.304	0.306	0.277	0.280	0.269	0.274
Dow Jones	0.037	0.0365	0.316	0.277	0.223	0.231	0.215	0.200	0.185	0.183	0.169	0.176
FTSE	0.470	0.461	0.456	0.283	0.322	0.279	0.346	0.206	0.336	0.240	0.319	0.278
NASDAQ	0.424	0.399	0.498	0.237	0.283	0.233	0.404	0.175	0.388	0.200	0.364	0.218
Smallcap	0.448	0.292	1.035	0.189	0.398	0.157	0.981	0.143	0.979	0.144	0.970	0.143
NYSE	0.318	0.245	0.470	0.144	0.277	0.148	0.418	0.107	0.413	0.121	0.404	0.136
DAX	0.903	0.800	1.142	0.559	0.639	0.552	0.992	0.501	0.961	0.580	0.920	0.697
Apple	3.736	3.535	1.853	2.100	2.069	1.816	1.380	1.787	1.305	1.818	1.280	1.749
Microsoft	2.729	3.388	1.600	1.999	1.156	1.898	0.981	1.335	0.766	1.084	0.664	0.973
AXP	2.350	2.13	2.142	1.500	1.456	1.264	1.726	1.065	1.759	0.984	1.625	0.928

One can see from Table 4 and 5 how for a wide range of datasets, starting point and forecasting horizon the time-varying AMSE is considerably smaller than that for the time-constant version. We believe, even if prediction and forecasting is more important from an economist's perspective, this POOS analysis provides a strong motivation to choose a time-varying model over a time-constant one.

5.3.2. *One-step ahead forecasting and semi-timevarying models from inference.* We use the following one-step ahead $AMSE_1$ used by [44] to validate our models

Table 5

Forecasting volatility: Short range forecast comparison of time-varying and time-constant model for startpoint at 1000. TV and TC stands for time-varying and time-constant models

Data	ahead=25		ahead=50		ahead=75		ahead=100		ahead=150		ahead=200	
	TC	TV	TC	TV	TC	TV	TC	TV	TC	TV	TC	TV
USGBP	0.068	0.019	0.064	0.017	0.062	0.015	0.061	0.014	0.059	0.013	0.058	0.014
USCHF	4.344	4.009	1.796	1.584	1.074	0.891	0.799	0.624	0.559	0.392	0.419	0.278
USCAD	0.029	0.028	0.024	0.027	0.022	0.024	0.019	0.023	0.017	0.021	0.015	0.020
EURGBP	0.019	0.012	0.017	0.011	0.016	0.010	0.015	0.009	0.015	0.010	0.014	0.011
EURUSD	0.039	0.028	0.034	0.029	0.032	0.031	0.032	0.034	0.029	0.040	0.027	0.045
SP Merval	10.39	8.714	9.432	6.12	7.666	6.029	8,522	5.883	7.908	5.857	7.052	5.739
BSE	2478	2126	2798	1437	1506	949	2288	783	2263	771	2362	945.9
SP500 OlD	8.104	7.964	12.57	11.26	6.242	5.949	4.292	3.469	2.307	1.463	1.741	1.216
SP500	0.371	0.503	0.328	0.276	0.259	0.236	0.231	0.196	0.183	0.166	0.155	0.152
DowJones	0.484	0.509	0.351	0.400	0.263	0.335	0.210	0.292	0.167	0.267	0.145	0.238
FTSE	0.644	0.639	0.514	0.378	0.409	0.394	0.358	0.270	0.347	0.353	0.314	0.419
NASDAQ	0.519	0.505	0.482	0.345	0.332	0.377	0.340	0.267	0.312	0.338	0.284	0.365
Smallcap	0.457	0.374	0.671	0.243	0.338	0.243	0.558	0.229	0.488	0.232	0.433	0.206
NYSE	0.375	0.329	0.374	0.207	0.271	0.212	0.287	0.153	0.260	0.194	0.237	0.228
DAX	1.245	1.123	1.130	0.755	0.842	0.708	0.925	0.762	0.897	0.896	0.832	1.102
Apple	2.465	2.275	1.799	1.279	1.530	1.206	1.374	0.871	1.276	1.036	1.237	1.052
Microsoft	2.868	3.668	1.720	2.295	1.214	2.147	1.057	1.596	0.851	1.335	0.770	1.232
AXP	2.837	2.668	1.749	1.462	1.395	1.347	1.151	0.825	0.982	0.681	0.834	0.581

$$AMSE_1 = \frac{1}{n} \sum_{i=1}^{n} (X_i^2 - \hat{\sigma}^2(i/n))^2$$

where X_t is the realized observations (log-returns) and $\hat{\sigma}^2(\cdot)$ refers to the fitted model using ARCH(1,1) for foreign exchange datasets, and GARCH(1,1) for stock market indices. In each row we exhibit the best model in bold.

Note that, the examples exhibited here show that often the time-constant model has poor one-step ahead pseudo-out-of-sample (POOS) forecasting compared to their time-varying analogue. However, from Table 6, one can see in some of these time-varying models we have a subset of parameters not rejecting the hypothesis of time-constancy. We suspect that setting a subset of parameters to be time-constant and allowing the rest to vary over time can improve forecasting over both the models. One find of this analysis is that it is not always true but for some of the datasets such as USGBP, the semi-time varying model achieves the best forecasting rate. Note that it is easy to tailor and find time-constant fits that allow for even better forecasting rate but those models do not have proper confidence (in terms of closeness to the true model) for the already observed data. For the numbers in the above table on the semi-time-varying column, we kept the time-varying coefficients as they are and searched for the best (in terms of AMSE) constant for the time-constant coefficients among the horizontal lines that fit within the bands entirely. Here we would

TABLE 6
Forecasting volatility: Choosing between asymptotically 95% correct time-varying, semi-time-varying and time-constant models

Index (opt $\hat{b}_n$)	Time-varying	Time-constant	Time-constant coefficients	Semi-timevarying model
USGBP (0.26)	0.5701109	0.5956046	α_1	**0.5697231**
USCHF (0.24)	**45.42274**	45.61378	None	NA
USCAD (0.22)	**0.1604907**	0.1672808	α_1	0.1616965
EURGBP (0.34)	**0.778459**	0.789287	α_1	0.7837281
EURUSD (0.23)	**0.3234517**	0.3399221	None	NA
SP Merval (0.36)	**60.62419**	62.02226	α_1, β_1	61.58843
SP500 old (0.29)	**26.48272**	26.12195	None	NA
SP500 recent (0.22)	3.54	**3.527918**	α_1, β_1	3.529355
Wilshire 5k (0.35)	**1.943517**	1.964926	$\alpha_0, \alpha_1, \beta_1$	1.972347
FTSE (0.45)	**3.650331**	3.687818	None	NA
NASDAQ (0.405)	**5.4258546**	5.446805	α_1, β_1	5.611163
Smallcap (0.35)	**12.36647**	12.56344	α_1, β_1	12.53796
NYSE (0.24)	**4.830424**	4.841198	$\alpha_0, \beta_1, \beta_1$	5.287622
DAX (0.44)	**9.908042**	9.987583	None	NA
Apple (0.27)	48.06839	**46.4968**	None	NA
Microsoft (0.39)	**38.08752**	38.33715	None	NA
AXP (0.28)	**37.39755**	37.90381	α_1, β_1	38.1988

like to also put a word of caution as the semi-timevarying analysis is somewhat adhoc and one can re-run the optimization by fitting only a proper subset as time-varying and the rest as time-constant. We have checked this with multiple of the above datasets and the AMSE were not too different from that reported above. Finally, the major takeaway from this analysis remains how our time-varying fit, albeit not meant for prediction and only for building simultaneous confidence interval can achieve better forecasting than the corresponding time-constant fits and often can lead to also consider models which has only a subset of coefficients time-varying.

References.

[1] ANDREOU, E. and GHYSELS, E. (2006). Monitoring disruptions in financial markets. *J. Econometrics* **135** 77–124. MR2328397

[2] ANDREWS, D. W. K. (1993). Tests for parameter instability and structural change with unknown change point. *Econometrica* **61** 821–856. MR1231678

[3] BAI, J. (1997). Estimation of a change point in multiple regression models. *The Review of Economics and Statistics* **79** 551–563.

[4] BILLINGSLEY, P. (1999). *Convergence of probability measures*, second ed. *Wiley Series in Probability and Statistics: Probability and Statistics*. John Wiley & Sons, Inc., New York A Wiley-Interscience Publication. MR1700749

[5] BOLLERSLEV, T. (1986). Generalized autoregressive conditional heteroskedasticity. *J. Econometrics* **31** 307–327. MR853051

[6] BROWN, R. L., DURBIN, J. and EVANS, J. M. (1975). Techniques for testing the constancy of regression relationships over time. *J. Roy. Statist. Soc. Ser. B* **37** 149–192. With discussion by D. R. Cox, P. R. Fisk, Maurice Kendall, M. B. Priestley, Peter C. Young, G. Phillips, T. W. Anderson, A. F. M. Smith,

M. R. B. Clarke, A. C. Harvey, Agnes M. Herzberg, M. C. Hutchison, Mohsin S. Khan, J. A. Nelder, Richard E. Quant, T. Subba Rao, H. Tong and W. G. Gilchrist and with reply by J. Durbin and J. M. Evans. MR0378310

[7] BURKHOLDER, D. L. (1988). Sharp inequalities for martingales and stochastic integrals. *Astérisque* **157-158** 75–94. Colloque Paul Lévy sur les Processus Stochastiques (Palaiseau, 1987). MR976214

[8] CAI, Z. (2007). Trending time-varying coefficient time series models with serially correlated errors. *J. Econometrics* **136** 163–188. MR2328589

[9] CHEN, J. and GUPTA, A. K. (1997). Testing and locating variance changepoints with application to stock prices. *Journal of the American Statistical association* **92** 739–747.

[10] CHOW, G. C. (1960). Tests of equality between sets of coefficients in two linear regressions. *Econometrica* **28** 591–605. MR0141193

[11] DAHLHAUS, R. (2011). Locally Stationary Processes. *Handbook of Statistics*.

[12] DAHLHAUS, R. and POLONIK, W. (2009). Empirical spectral processes for locally stationary time series. *Bernoulli* **15** 1–39. MR2546797

[13] DAHLHAUS, R., RICHTER, S. and WU, W. B. (2017). Towards a general theory for non-linear locally stationary processes. *ArXiv e-prints: 1704.02860*.

[14] DAHLHAUS, R. and SUBBA RAO, S. (2006). Statistical inference for time-varying ARCH processes. *Ann. Statist.* **34** 1075–1114. MR2278352

[15] ENGLE, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica* **50** 987–1007. MR666121

[16] ENGLE, R. F. and RANGEL, J. G. (2005). The spline garch model for unconditional volatility and its global macroeconomic causes.

[17] FAN, J. and ZHANG, W. (1999). Statistical estimation in varying coefficient models. *Ann. Statist.* **27** 1491–1518. MR1742497

[18] FAN, J. and ZHANG, W. (2000). Simultaneous Confidence Bands and Hypothesis Testing in Varying-coefficient Models. *Scandinavian Journal of Statistics* **27** 715–731.

[19] FRANCQ, C. and ZAKOÏAN, J.-M. (2004). Maximum likelihood estimation of pure GARCH and ARMA-GARCH processes. *Bernoulli* **10** 605–637. MR2076065

[20] FRYZLEWICZ, P., SAPATINAS, T. and SUBBA RAO, S. (2008a). Normalized least-squares estimation in time-varying ARCH models. *Ann. Statist.* **36** 742–786. MR2396814

[21] FRYZLEWICZ, P., SAPATINAS, T. and SUBBA RAO, S. (2008b). Normalized least-squares estimation in time-varying ARCH models. *Ann. Statist.* **36** 742–786. MR2396814

[22] GIACOMETTI, R., BERTOCCHI, M., RACHEV, S. T. and FABOZZI, F. J. (2012). A comparison of the Lee–Carter model and AR–ARCH model for forecasting mortality rates. *Insurance: Mathematics and Economics* **50** 85–93.

[23] HARDLE, W. (1986). A note on jackknifing kernel regression function estimators (corresp.). *IEEE transactions on information theory* **32** 298–300.

[24] HOOVER, D. R., RICE, J. A., WU, C. O. and YANG, L.-P. (1998). Nonparametric smoothing estimates of time-varying coefficient models with longitudinal data. *Biometrika* **85** 809–822. MR1666699

[25] HUANG, J. Z., WU, C. O. and ZHOU, L. (2004). Polynomial spline estimation and inference for varying coefficient models with longitudinal data. *Statist. Sinica* **14** 763–788. MR2087972

[26] JAMES CHU, C.-S. (1995). Detecting parameter shift in GARCH models. *Econometric Reviews* **14** 241–266.

[27] KARMAKAR, S. (2018). Asymptotic Theory for Simultaneous Inference Under Dependence Technical Report, University of Chicago.

[28] KARMAKAR, S. and WU, W. B. (2020). Optimal gaussian approximation for multiple time series. *To appear in Statistica Sinica arXiv preprint arXiv:2001.10164*.

[29] KIM, S., CHO, S. and LEE, S. (2000). On the cusum test for parameter changes in GARCH (1, 1) models. *Communications in Statistics-Theory and Methods* **29** 445–462.

[30] KOKOSZKA, P., LEIPUS, R. et al. (2000). Change-point estimation in ARCH models. *Bernoulli* **6**

513–539.

[31] KULPERGER, R., YU, H. et al. (2005). High moment partial sum processes of residuals in GARCH models and their applications. *The Annals of Statistics* **33** 2395–2422.

[32] LEYBOURNE, S. J. and MCCABE, B. P. M. (1989). On the distribution of some test statistics for coefficient constancy. *Biometrika* **76** 169–177. MR991435

[33] LIN, C.-F. J. and TERÄSVIRTA, T. (1999). Testing parameter constancy in linear models against stochastic stationary parameters. *J. Econometrics* **90** 193–213. MR1703341

[34] LIN, S.-J., YANG, J. et al. *Testing shifts in financial models with conditional heteroskedasticity: an empirical distribution function approach.*

[35] LIN, D. Y. and YING, Z. (2001). Semiparametric and nonparametric regression analysis of longitudinal data. *J. Amer. Statist. Assoc.* **96** 103–126. With comments and a rejoinder by the authors. MR1952726

[36] LING, S. and MCALEER, M. (2002). Necessary and sufficient moment conditions for the GARCH(r, s) and asymmetric power GARCH(r, s) models. *Econometric Theory* **18** 722–729. MR1906332

[37] MIKOSCH, T. and STĂRICĂ, C. (2004). Nonstationarities in financial time series, the long-range dependence, and the IGARCH effects. *The Review of Economics and Statistics* **86** 378–390.

[38] NABEYA, S. and TANAKA, K. (1988). Asymptotic theory of a test for the constancy of regression coefficients against the random walk alternative. *Ann. Statist.* **16** 218–235. MR924867

[39] NYBLOM, J. (1989). Testing for the constancy of parameters over time. *J. Amer. Statist. Assoc.* **84** 223–230. MR999682

[40] PALM, F. C. (1996). GARCH models of volatility. In *Statistical methods in finance. Handbook of Statist.* **14** 209–240. North-Holland, Amsterdam. MR1602132

[41] PLOBERGER, W., KRÄMER, W. and KONTRUS, K. (1989). A new test for structural stability in the linear regression model. *J. Econometrics* **40** 307–318. MR994952

[42] RAMSAY, J. O. and SILVERMAN, B. W. (2005). *Functional data analysis*, second ed. *Springer Series in Statistics.* Springer, New York. MR2168993

[43] RICHTER, S. and DAHLHAUS, R. (2017). Cross validation for locally stationary processes. *ArXiv e-prints: 1705.10046.*

[44] ROHAN, N. (2013). A time varying GARCH (p, q) model and related statistical inference. *Statistics & Probability Letters* **83** 1983–1990.

[45] ROHAN, N. and RAMANATHAN, T. V. (2013). Nonparametric estimation of a time-varying GARCH model. *J. Nonparametr. Stat.* **25** 33–52. MR3039969

[46] STARICA, C. (2003). Is GARCH (1, 1) as good a model as the accolades of the Nobel prize would imply? *Available at SSRN 637322.*

[47] STĂRICĂ, C. and GRANGER, C. (2005). Nonstationarities in stock returns. *The Review of Economics and Statistics* **87** 503–522.

[48] VOGT, M. et al. (2012). Nonparametric regression for locally stationary time series. *The Annals of Statistics* **40** 2601–2633.

[49] WITTING, H. and MÜLLER-FUNK, U. (1995). *Mathematische Statistik. II.* B. G. Teubner, Stuttgart Asymptotische Statistik: parametrische Modelle und nichtparametrische Funktionale. [Asymptotic statistics: parametric models and nonparametric functionals]. MR1363716

[50] WU, W. B. (2005). Nonlinear system theory: another look at dependence. *Proc. Natl. Acad. Sci. USA* **102** 14150–14154 (electronic). MR2172215

[51] WU, W. B. and MIN, W. (2005). On linear processes with dependent innovations. *Stochastic Processes and their Applications* **115** 939 - 958.

[52] WU, W. B. and SHAO, X. (2004). Limit theorems for iterated random functions. *J. Appl. Probab.* **41** 425–436. MR2052582

[53] WU, W. B. and ZHOU, Z. (2011). Gaussian approximations for non-stationary multiple time series. *Statist. Sinica* **21** 1397–1413. MR2827528

[54] ZHANG, W., LEE, S.-Y. and SONG, X. (2002). Local polynomial fitting in semivarying coefficient model. *J. Multivariate Anal.* **82** 166–188. MR1918619

[55] Zhang, T. and Wu, W. B. (2012). Inference of time-varying regression models. *Ann. Statist.* **40** 1376–1402. MR3015029

[56] Zhang, T. and Wu, W. B. (2015). Time-varying nonlinear regression models: nonparametric estimation and model selection. *Ann. Statist.* **43** 741–768. MR3319142

[57] Zhang, D. and Wu, W. B. (2017). Gaussian Approximation for High Dimensional Time Series. *Ann. Statist.* **45** 1895–1919.

[58] Zhou, Z. and Wu, W. B. (2009). Local linear quantile estimation for nonstationary time series. *Ann. Statist.* **37** 2696–2729. MR2541444

[59] Zhou, Z. and Wu, W. B. (2010). Simultaneous inference of linear models with time varying coefficients. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **72** 513–531. MR2758526

Supplement: This material ([doi]COMPLETED BY THE TYPESETTER) contains the proofs of the results in the paper as well as the proofs of the examples.

6. Technical tools.

6.1. *The functional dependence measure.* To state the structure of dependence we use throughout the paper, we introduce a functional dependence measure on the underlying process using the idea of coupling as done in Wu [50]. Assume that a stationary process Z_i has mean 0, $Z_i \in \mathcal{L}_q, q > 0$ and it admits the causal representation

$$(6.1) \qquad Z_i = J(\zeta_i, \zeta_{i-1}, \ldots).$$

Suppose that $(\zeta_i^*)_{i \in \mathbb{Z}}$ is an independent copy of $(\zeta_i)_{i \in \mathbb{Z}}$. For some random variable Z, let $\|Z\|_q := (\mathbb{E}|Z|^q)^{1/q}$ denote the $\mathcal{L}_q$-norm of Z. For $j \geq 0$, define the functional dependence measure

$$(6.2) \qquad \delta_q^Z(i) = \|Z_i - Z_i^*\|_q,$$

where $\mathcal{F}_i^*$ is a coupled version of $\mathcal{F}_i$ with ζ_0 in $\mathcal{F}_i$ replaced by ζ_0^*,

$$(6.3) \qquad \mathcal{F}^* = (\zeta_i, \zeta_{i-1}, \cdots, \zeta_1, \zeta_0^*, \zeta_{-1}, \zeta_{-2}, \cdots),$$

and $Z_i^* = J(\mathcal{F}_i^*)$. Note that $\delta_q^Z(i)$ measures the dependence of Z_i on ζ_0 in terms of the qth moment. The tail cumulative dependence measure $\Delta_q^Z(j)$ for $j \geq 0$ is defined as

$$(6.4) \qquad \Delta_q^Z(j) = \sum_{i=j}^{\infty} \delta_q^Z(i).$$

6.2. *The class $\mathcal{H}(M_y, M_x, \chi, \bar{C})$.* To prove uniform convergence of L_{n,b_n}^c and its derivatives w.r.t. θ, we require ℓ to be Lipschitz continuous in direction of θ and to grow at most polynomially in direction of $z = (y, x)$, where the degree is measured by integers $M_y, M_x \geq 1$. We will therefore ask ℓ and its derivatives to be in the class $\mathcal{H}(M_y, M_x, \chi, \bar{C})$ which is defined as follows: Let $\chi = (\chi_i)_{i=1,2,\ldots}$ be a sequence of nonnegative real numbers with $|\chi|_1 := \sum_{i=1}^{\infty} \chi_i < \infty$, and $\bar{C} > 0$ be some constant. Define $|x|_{\chi,1} := \sum_{i=1}^{\infty} \chi_i |x_i|$. Put $\hat{\chi} = (1, \chi)$, and for nonnegative integers d_x, d_y, define the 'polynomial rest'

$$R_{d_y, d_x}(z) := \sum_{\substack{k=0 \\ k+l \leq \max\{d_x, d_y\}}}^{d_y} \sum_{l=0}^{d_x} |y|^k |x|_{\chi,1}^l.$$

A function $g : \mathbb{R} \times \mathbb{R}^{\mathbb{N}} \times \Theta \to \mathbb{R}$ is in $\mathcal{H}(M_y, M_x, \chi, \bar{C})$ if $\sup_{\theta \in \Theta} |g(0, \theta)| \leq \bar{C}$,

$$\sup_z \sup_{\theta \neq \theta'} \frac{|g(z, \theta) - g(z, \theta')|}{|\theta - \theta'|_1 R_{M_y, M_x}(z)} \leq \bar{C}$$

and

$$\sup_{\theta} \sup_{z \neq z'} \frac{|g(z, \theta) - g(z', \theta)|}{|z - z'|_{\hat{\chi}, 1} \cdot \{R_{M_y-1, M_x-1}(z) + R_{M_y-1, M_x-1}(z')\}} \leq \bar{C}.$$

If g is vector- or matrix-valued, $g \in \mathcal{H}(M_y, M_x, \chi, \bar{C})$ means that every component of g is in $\mathcal{H}(M_y, M_x, \chi, \bar{C})$. In case of time series it often holds that $M = M_x = M_y$, which allows to use a simplified version $R_{M_y, M_x}(z) = 1 + |z|_{\hat{\chi}, 1}^M$.

7. Assumptions. In this paper, we prove weak Bahadur representations and construct simultaneous confidence bands for $\hat{\theta}_{b_n}(\cdot)$ and $\widehat{\theta'_{b_n}}(\cdot)$. Clearly, more smoothness assumptions on $\theta(\cdot)$ and ℓ are needed to prove results for the latter one. To do so, we introduce a more general set of assumptions which is implied by either Assumption 2.1 or 2.2. This is done in Section 9 below.

7.1. *Case 1.* In the following, we will assume the existence of measurable functions H, G such that $\tilde{Y}_i(t) = H(t, \mathcal{F}_i) \in \mathbb{R}$ and $\tilde{X}_i(t) = G(t, \mathcal{F}_i) \in \mathbb{R}^{\mathbb{N}}$ are well-defined for all $t \in [0, 1]$. These processes will serve as stationary approximations of Y_i, X_i if $|i/n - t| \ll 1$. For brevity, define $\tilde{Z}_i(t) := (\tilde{Y}_i(t), \tilde{X}_i(t)^{\mathsf{T}})^{\mathsf{T}}$ and $Z_i := (Y_i, X_i^{\mathsf{T}})^{\mathsf{T}}$. The constant $r \geq 2$ in the following assumption is connected to the number of moments that are assumed for Z_i (cf. (A5) and (A7)), while $\gamma > 1$ is a measure of decay of the dependence which is present in the model.

A slightly different set of assumptions (Assumption 7.3, 7.4) which is specifically designed for conditional heteroscedastic models, leading to weaker moment assumptions, is given below. All theoretical results in this paper also hold under this set of assumptions.

ASSUMPTION 7.1. *Suppose that for some $r \geq 2$ and some $\gamma > 1$,*

(A1) *(Smoothness in θ-direction) ℓ is twice continuously differentiable w.r.t. θ. It holds that $\ell, \nabla_\theta \ell, \nabla_\theta^2 \ell \in \mathcal{H}(M_y, M_x, \chi, \bar{C})$ for some $M_y, M_x \geq 1$, $\bar{C} > 0$ and $\chi = (\chi_i)_{i=1,2,\ldots}$ with $\chi_i = O(i^{-(1+\gamma)})$.*

(A2) *(Assumptions on unknown parameter curve) Θ is compact and for all $t \in [0, 1]$, $\theta(t)$ lies in the interior of Θ. Each component of $\theta(\cdot)$ is in $C^3[0, 1]$.*

(A3) *(Correct model specification) For all $t \in [0, 1]$, the function $\theta \mapsto L(t, \theta) := \mathbb{E}\ell(\tilde{Z}_0(t), \theta)$ is uniquely minimized by $\theta(t)$.*

(A4) *The eigenvalues of the matrices*

$$
\begin{align}
V(t) &= \mathbb{E}\nabla_\theta^2 \ell(\tilde{Z}_0(t), \theta(t)), \tag{7.1}\\
I(t) &= \mathbb{E}[\nabla_\theta \ell(\tilde{Z}_0(t), \theta(t)) \cdot \nabla_\theta \ell(\tilde{Z}_0(t), \theta(t))^{\mathsf{T}}], \tag{7.2}\\
\Lambda(t) &= \sum_{j \in \mathbb{Z}} \mathbb{E}[\nabla_\theta \ell(\tilde{Z}_0(t), \theta(t)) \cdot \nabla_\theta \ell(\tilde{Z}_j(t), \theta(t))^{\mathsf{T}}], \tag{7.3}
\end{align}
$$

are bounded from below by some $\lambda_0 > 0$, uniformly in t.

(A5) (Stationary approximation) Let $M = \max\{M_x, M_y\}$. There exist $C_A, C_B, D > 0$ such that for all $n \in \mathbb{N}$, $i = 1, \ldots, n$, $t, t' \in [0, 1]$, $j \in \mathbb{N}$:

$$\max\{\|Y_i\|_{rM},\ \|\tilde{Y}_0(t)\|_{rM},\ \|X_{ij}\|_{rM},\ \|\tilde{X}_{0j}(t)\|_{rM}\} \le D,$$

and

$$\|X_{ij} - \tilde{X}_{ij}(i/n)\|_{rM} \le C_A n^{-1}, \qquad \|\tilde{X}_{0j}(t) - \tilde{X}_{0j}(t')\|_{rM} \le C_B|t - t'|,$$

and <u>either</u>

$$(7.4) \qquad \|Y_i - \tilde{Y}_i(i/n)\|_{rM} \le C_A n^{-1}, \qquad \|\tilde{Y}_0(t) - \tilde{Y}_0(t')\|_{rM} \le C_B|t - t'|$$

<u>or</u> (with χ from (A1))

$$(7.5) \qquad\qquad\qquad \sup_{x \ne x'} \frac{\|F_i(x, \theta) - F_i(x', \theta)\|_{M_y}}{|x - x'|_{\chi,1}} < \infty.$$

(A6) (Negligibility of the truncation) For all i, j: $|X_{ij}^c| \le |X_{ij}|$. For $1 \le j \le i$, $X_{ij} = X_{ij}^c$.

(A7) (Weak dependence) It holds that $\sup_{t \in [0,1]} \delta_{rM}^{\tilde{X}(t)}(k) = O(k^{-(1+\gamma)})$ and <u>either</u> (7.5) <u>or</u> $\sup_{t \in [0,1]} \delta_{rM}^{\tilde{Y}(t)}(k) = O(k^{-(1+\gamma)})$ holds.

Note that (A2), (A3) and (A4) are typical assumptions in M-estimation theory to guarantee convergence of the estimator towards the correct parameter and to use Taylor expansions and bias expansions. The condition on L in (A3) directly implies $0 = \nabla_\theta L(t, \theta(t)) = \mathbb{E}\nabla_\theta \ell(\tilde{Z}_0(t), \theta(t))$ under the imposed smoothness conditions, which will be used in the proofs. In many special cases in time series analysis (cf. Example 9.1), it may even occur that $\nabla_\theta \ell(\tilde{Z}_0(t), \theta(t))$ is a martingale difference sequence or at least an uncorrelated sequence. In these cases, $\Lambda(t) = I(t)$ such that the verification of (A4) is simplified.

Asking the objective function ℓ to be twice continuously differentiable w.r.t. θ as done in (A1) is a typical condition and is needed to use Taylor expansions. We additionally ask ℓ and its derivatives w.r.t. θ to be in $\mathcal{H}(M_y, M_x, \chi, \tilde{C})$. This is exploited in two ways: It allows quantification of the order of dependence of $\ell(Y_i, X_i, \theta)$ based on the dependence of X_i, Y_i, and it allows to deal with local stationarity by replacing X_i, Y_i by its stationary counterparts. In this context, we especially need a decay condition on the coefficients x_i which appear in ℓ. This decay is quantified by the sequence $\chi = (\chi_i)_{i \in \mathbb{N}}$. We use this rate to show that the observed truncated values X_i^c are negligible compared to X_i and that the overall dependence of $\ell(Y_i, X_i, \theta)$ has the same order as the original sequences Y_i, X_i (cf. (A7)). Lastly, condition (A1) implicitly implies continuity of the matrices appearing in (A4) such that it is enough to show pointwise positive definiteness.

To eliminate bias terms, we state (A5) which asks for smoothness of the processes X_i, Y_i in time direction and the existence of a stationary approximation. The sense of local stationarity is similar to what is used in [14, 48] etc. and a detailed discussion can be found

in [11]. Here we consider two different cases. The case (7.5) is dedicated to general linear models which may have discretely distributed observations Y_i and thus would not fulfill a condition like (7.4) for $rM \geq 2$. To prove central limits theorems and to use strong Gaussian approximations, we need a weak dependence assumption which is given in (A7). Let us emphasize the fact that all conditions besides (A5) are formulated for the stationary approximation $\tilde{Z}_i(t) = (\tilde{Y}_i(t), \tilde{X}_i(t))$ which in general allows easier verification and the possibility to use earlier results obtained for stationary settings.

To prove a typical second-order bias decomposition for $\widehat{\theta}'_{b_n}(t)$, we need that the stationary approximations $\tilde{Z}_i(t)$ are differentiable w.r.t. t. The concept of derivative processes in the context of locally stationary processes was originally introduced in Dahlhaus [11] and Dahlhaus and Subba Rao [14] and formalized in Dahlhaus, Richter and Wu [13] especially for processes with Markov structure.

ASSUMPTION 7.2 (Differentiability assumptions). *Suppose that there exist* $M'_y, M'_x \geq 2$ *such that* $M' := \max\{M'_x, M'_y\}$ *fufills* $M' \leq rM$ *and*

(B1) $\theta(\cdot) \in C^4[0,1]$.
(B2) $\nabla^2_\theta \ell(z,\theta)$ *is continuously differentiable. It holds that* $\nabla^3_\theta \ell \in \mathcal{H}(M'_y, M'_x, \chi, \bar{C})$, *and for all* $l \in \mathbb{N}_0$, $\partial_{z_l} \nabla^2_\theta \ell \in \mathcal{H}(M'_y - 1, M'_x - 1, \chi', \bar{C}\hat{\chi}_l)$ *with some absolutely summable sequence* $\chi' = (\chi'_i)_{i=1,2,\ldots}$.
(B3) $t \mapsto \tilde{Z}_0(t)$ *is continuously differentiable and* $\sup_{t \in [0,1]} \sup_{j \in \mathbb{N}_0} \|\partial_t \tilde{Z}_{0j}(t)\|_{M'} \leq D$,

$$\sup_{j \in \mathbb{N}_0} \sup_{t \neq t'} \frac{\|\partial_t \tilde{Z}_{0j}(t) - \partial_t \tilde{Z}_{0j}(t')\|_{M'}}{|t - t'|} \leq C_B.$$

Note that the condition $\partial_{x_l} \nabla^2_\theta \ell \in \mathcal{H}(M'_y, M'_x, \chi', \bar{C}\chi_l)$ asks $\nabla^2_\theta \ell$ to be dependent on x_l with a factor of at most χ_l which is a stronger condition than the corresponding condition on $\nabla^2_\theta \ell$ in (A1).

7.2. *Case 2: tvGARCH.* Assumptions 7.1, 7.2 are formulated as general as possible to cover a lot of different models. However in specific situations, the conditions therein may be too strong. Because of that, let us introduce the more general class $\mathcal{H}_s(M_y, M_x, \chi, \bar{C})$ for $s \geq 0$: A function $g : \mathbb{R} \times \mathbb{R}^\mathbb{N} \times \Theta \to \mathbb{R}$ is in $\mathcal{H}_s(M_y, M_x, \chi, \bar{C})$ if $\sup_{\theta \in \Theta} |g(0, \theta)| \leq \bar{C}$,

$$\sup_z \sup_{\theta \neq \theta'} \frac{|g(z, \theta) - g(z, \theta')|}{|\theta - \theta'|_1 R_{M_y, M_x}(z)^{1+s}} \leq \bar{C}$$

and

$$\sup_\theta \sup_{z \neq z'} \frac{|g(z, \theta) - g(z', \theta)|}{|z - z'|_{\hat{\chi}, 1} \cdot \{R_{M_y-1, M_x-1}(z)^{1+s} + R_{M_y-1, M_x-1}(z')^{1+s}\}} \leq \bar{C}.$$

Obviously, $\mathcal{H} = \mathcal{H}_0$.

To make use of independencies occuring in the analysis, let us introduce the class $\mathcal{H}_{s,\iota}^{mult}(M_y, M_x, \chi, \bar{C})$ for $s \geq 0$ which consists of functions $g : \mathbb{R} \times \mathbb{R}^{\mathbb{N}} \times \Theta$ such that $\sup_{\theta \in \Theta} \frac{|g(y,0,\theta)|}{1+|y|^{M_y}} \leq \bar{C}$ and

$$\sup_{|\theta-\tilde{\theta}|_1<\iota} \sup_{y} \sup_{x\neq x'} \frac{|g(y,x,\theta) - g(y,x',\theta)|}{|x-x'|_{\chi,1}(R_{M_y-1,M_x-1}(1,x)^{1+s} + R_{M_y-1,M_x-1}(1,x')^{1+s})(1+|y|^M)} \leq \bar{C},$$

$$\sup_{x,y} \sup_{\theta\neq\theta',|\theta-\tilde{\theta}|_1<\iota,|\theta'-\tilde{\theta}|_1<\iota} \frac{|g(y,x,\theta) - g(y,x,\theta')|}{|\theta-\theta'|_1 R_{M_y,M_x}(1,x)^{1+s}(1+|y|^{M_y})} \leq \bar{C}.$$

Let $|x|_{\chi,s} := (\sum_{j=1}^{\infty} \chi_j |x|^s)^{1/s}$.

ASSUMPTION 7.3 (Heteroscedastic recursively defined time series case). *Let ζ_i, $i \in \mathbb{Z}$ be an i.i.d. sequence. Assume that the stationary approximation $\tilde{Y}_i(t)$ of Y_i evolves according to*

$$\tilde{Y}_i(t) = F(\tilde{X}_i(t), \theta(t), \zeta_i)$$

where $\tilde{X}_i(t) = (\tilde{Y}_{i-1}(t), \tilde{Y}_{i-2}(t), ...)$.
Let

$$\tilde{\ell}_{\tilde{\theta}}(y, x, \theta) := \ell(F(x, \tilde{\theta}, y), x, \theta).$$

Suppose that for some $r \geq 2$ and $\gamma > 1$,

(A1') *ℓ is twice continuously differentiable w.r.t. θ. There exists $M \geq 1$ such that for each $s > 0$, there exist $\chi^{(s)} = (\chi_j^{(s)})_{j=1,2,...}$ with $\chi_j^{(s)} = O(j^{-(1+\gamma)})$ and $\bar{C}^{(s)} > 0$, such that*

- *$\ell, \nabla_\theta \ell, \nabla_\theta^2 \ell \in \mathcal{H}_s(2M, 2M, \chi^{(s)}, \bar{C}^{(s)})$.*

-

  (7.6)

$$\sup_{\theta} \sup_{z\neq z'} \frac{|\ell(z,\theta) - \ell(z',\theta)|}{|z-z'|_{\hat{\chi}^{(s)},s}^s(R_{M,M}(z) + R_{M,M}(z')) + |z-z'|_{\hat{\chi}^{(s)},1}(R_{M-1,M-1}(z)^{1+s} + R_{M-1,M-1}(z')^{1+s})} \leq \bar{C}^{(s)}.$$

- *There exists $\iota > 0$ such that $\nabla_\theta \tilde{\ell}, \nabla_\theta^2 \tilde{\ell} \in \mathcal{H}_{s,\iota}^{mult}(M, M, \chi^{(s)}, \bar{C}^{(s)})$*

(A2') *(A2) holds,*
(A3') *(A3) holds,*
(A4') *(A4) holds,*
(A5') *(A5) holds with (7.4) and $\|\zeta_0\|_{rM} \leq D$.*
(A6') *$X_i^c = (Y_{i-1}, Y_{i-2}, ..., Y_1, 0, 0, ...)$*
(A7') *$\sup_{t\in[0,1]} \delta_{rM}^{\tilde{Y}(t)}(k) = O(\rho^k)$ with some $\rho \in (0,1)$.*

ASSUMPTION 7.4 (Heteroscedastic recursively defined time series case). *Suppose that there exists $M' \geq 2$ such that $M' \leq rM$ and for all $s > 0$ there exist absolutely summable $(\chi')^{(s)} = ((\chi')_j^{(s)})_{j=1,2,...}$ and $\iota > 0$ such that*

(B1') (B1) holds,

(B2') $\nabla^2_\theta \ell$ is continuously differentiable. It holds that $\nabla^3_\theta \tilde{\ell} \in \mathcal{H}^{mult}_{s,\iota}(M', M', \chi^{(s)}, \bar{C}^{(s)})$, and for all $i \in \mathbb{N}_0$, $\partial_{x_l} \nabla^2_\theta \tilde{\ell} \in \mathcal{H}^{mult}_{s,\iota}(M'-1, M'-1, (\chi')^{(s)}, \bar{C}^{(s)} \chi_l^{(s)})$.

(B3') (B3) holds.

8. Appendix: Proofs of the theorems. For $\eta = (\eta_1, \eta_2) \in \Theta \times (\Theta' \cdot b_n) =: E_n$, define

$$L^{\circ,c}_{n,b_n}(t, \eta) := (nb_n)^{-1} \sum_{i=1}^{n} K_{b_n}(i/n - t)\ell(Z_i^c, \eta_1 + \eta_2(i/n - t)b_n^{-1})$$

and $\hat{L}^\circ_{n,b_n}$, L°_{n,b_n} similarly as $L^{\circ,c}_{n,b_n}$ but with Z_i^c replaced by $\tilde{Z}_i(i/n)$ or Z_i, respectively. We define $\eta_{b_n}(t) = (\theta(t)^\mathsf{T}, b_n \theta'(t)^\mathsf{T})^\mathsf{T}$ as the value which should be estimated by $\hat{\eta}_{b_n}(t) = (\hat{\theta}_{b_n}(t)^\mathsf{T}, b_n \widehat{\theta'}_{b_n}(t)^\mathsf{T})^\mathsf{T}$, the minimizer of $L^\circ_{n,b_n}(t, \eta)$. In the proof of Theorem 8.1, it is shown that $L^\circ_{n,b_n}(t, \eta)$ converges to $L^\circ(t, \eta) := \int_{-1}^{1} K(x) L(t, \eta_1 + \eta_2 x) dx$. If $\chi \in \mathbb{R}^\mathbb{N}$, recall that $\hat{\chi} = (1, \chi) \in \mathbb{R}^{\mathbb{N}_0}$.

For $t \in (0,1)$ and $\eta \in E_n = \Theta \times (\Theta' \cdot b_n)$ and some Lipschitz continuous function $\hat{K}$ (Lipschitz constant $L_{\hat{K}}$) and compact support $[-1, 1]$ ($\hat{K}$ bounded by $|\hat{K}|_\infty$), define $\hat{K}_{b_n}(\cdot) := \hat{K}(\cdot/b_n)$ and

(8.1)

$$G_n(t, \eta) := (nb_n)^{-1} \sum_{i=1}^{n} \hat{K}_{b_n}(i/n - t) \cdot \{g(Z_i, \eta_1 + \eta_2(i/n - t)b_n^{-1}) - \mathbb{E}g(Z_i, \eta_1 + \eta_2(i/n - t)b_n^{-1})\}.$$

Let $G_n^c(t, \eta)$, $\hat{G}_n(t, \eta)$ denote the same quantities but with Z_i replaced by Z_i^c or $\tilde{Z}_i(i/n)$, respectively.

8.1. *Proofs of Section 3.*

THEOREM 8.1. *Fix $t \in (0,1)$. Let Assumption 7.1 hold with $r = 2$. Assume that $nb_n \to \infty$, $b_n \to 0$.*

(i) *(Consistency) It holds that $\hat{\theta}_{b_n}(t) - \theta(t) = o_\mathbb{P}(1)$.*
   If additionally $nb_n^3 \to \infty$, it holds that $\widehat{\theta'}_{b_n}(t) - \theta'(t) = o_\mathbb{P}(1)$.

Assume that $\sup_{j \in \mathbb{N}_0} \sup_{t \in [0,1]} \|Z_{0j}(t)\|_{(2+a)M} < \infty$ for some $a > 0$.

(ii) *If $nb_n^7 \to 0$, then*

(8.2) $$\sqrt{nb_n}(\hat{\theta}_{b_n}(t) - \theta(t) - b_n^2 \frac{\mu_{K,2}}{2} \theta''(t)) \xrightarrow{d} N(0, \sigma^2_{K,0} \cdot V(t)^{-1} I(t) V(t)^{-1}).$$

(iii) If additionally, Assumption 7.2 is fulfilled and $nb_n^9 \to 0$, then

$$
\begin{pmatrix} \sqrt{nb_n}(\hat{\theta}_{b_n}(t) - \theta(t) - b_n^2 \frac{\mu_{K,2}}{2} \theta''(t)) \\ \sqrt{nb_n^3}(\hat{\theta}'_{b_n}(t) - \theta'(t) - b_n^2 \frac{\mu_{K,4}}{2\mu_{K,2}} bias(t)) \end{pmatrix}
$$

$$
(8.3) \qquad \xrightarrow{d} \quad N\left(0, \sigma_{K,0}^2 \begin{pmatrix} 1 & 0 \\ 0 & \mu_{K,2}^{-2} \end{pmatrix} \otimes \{V(t)^{-1}I(t)V(t)^{-1}\}\right),
$$

where $bias(t) = \frac{1}{3}\theta^{(3)}(t) + V(t)^{-1}\mathbb{E}[\partial_t \nabla_\theta^2 \ell(\tilde{Z}_0(t), \theta(t))]\theta''(t)$.

The results hold true if instead of Assumption 7.1, 7.2, Assumption 7.3, 7.4 with some $r > 2$ is assumed.

PROOF OF THEOREM 8.1. The proof is similar to the proof of Theorems 5.2 and 5.4 in [13].

(i) Fix $t \in (0, 1)$. By Lemma 8.10(ii) applied to ℓ, we have

$$
\sup_{\eta \in E_n} |\hat{L}_{n,b_n}^\circ(t, \eta) - \mathbb{E}\hat{L}_{n,b_n}^\circ(t, \eta)| = o_{\mathbb{P}}(1).
$$

Applying Lemma 8.11 to ℓ, we obtain

$$
\sup_{\eta \in E_n} |\mathbb{E}\hat{L}_{n,b_n}^\circ(t, \eta) - L^\circ(t, \eta)| = O(b_n + (nb_n)^{-1}) = o(1),
$$

where $L^\circ(t, \eta) = \int_{-1}^1 K(x)L(t, \eta_1 + \eta_2 x)dx$. By Lemma 8.10(i), we have

$$
\Big\| \sup_{\eta \in E_n} |L_{n,b_n}^{\circ,c}(t, \eta) - \hat{L}_{n,b_n}^\circ(t, \eta)| \Big\|_1 = O((nb_n)^{-1}),
$$

and thus

$$
\sup_{\eta \in E_n} |L_{n,b_n}^{\circ,c}(t, \eta) - L^\circ(t, \eta)| = o_{\mathbb{P}}(1).
$$

By Lemma 8.3, $\eta \mapsto L^\circ(t, \eta)$ is Lipschitz continuous. Since $\theta(t)$ is the unique minimizer of $\theta \mapsto L(t, \theta)$, we conclude that $(\eta_1, \eta_2) = (\theta(t), 0)$ is the unique minimizer of $\eta \mapsto L^\circ(t, \eta)$. Since $\hat{\eta}_{b_n}(t) = (\hat{\theta}_{b_n}(t)^\mathsf{T}, b_n\hat{\theta}'_{b_n}(t)^\mathsf{T})^\mathsf{T}$ is a minimizer of $L_{n,b_n}^{\circ,c}(t, \eta)$, standard arguments yield

$$
(8.4) \qquad \hat{\eta}_{b_n}(t) = (\hat{\theta}_{b_n}(t)^\mathsf{T}, b_n\hat{\theta}'_{b_n}(t)^\mathsf{T})^\mathsf{T} = (\theta(t)^\mathsf{T}, 0)^\mathsf{T} + o_{\mathbb{P}}(1).
$$

We now show that $\hat{\theta}'_{b_n}(t) - \theta'(t) = o_{\mathbb{P}}(1)$ if $nb_n^3 \to \infty$. The following argumentation is also a preparation for the proof of (ii),(iii). By (8.4), we have that $\hat{\eta}_{b_n}(t)$ is in the interior of $\Theta \times (\Theta' \cdot b_n)$ with probability tending to 1 (since it converges to $(\theta(t)^\mathsf{T}, 0)$ in probability), thus $\nabla_\eta L_{n,b_n}^{\circ,c}(t, \hat{\eta}_{b_n}(t)) = 0$ with probability tending to 1. By a Taylor expansion we obtain

$$
(8.5) \qquad \hat{\eta}_{b_n}(t) - \eta_{b_n}(t) = -\left[\nabla_\eta^2 L_{n,b_n}^{\circ,c}(t, \bar{\eta}(t))\right]^{-1} \cdot \nabla_\eta L_{n,b_n}^{\circ,c}(t, \eta_{b_n}(t)),
$$

with some $\bar{\eta}(t) \in \Theta \times (\Theta' \cdot b_n)$ satisfying $|\bar{\eta}(t) - \eta_{b_n}(t)| \leq |\hat{\eta}_{b_n}(t) - \eta_{b_n}(t)|$. Let $V(t, \theta) :=$ $\mathbb{E}\nabla_\theta^2 \ell(\tilde{Z}_0(t), \theta)$. Since $g = \nabla_\theta^2 \ell \in \mathcal{H}(M_y, M_x, \chi, \bar{C})$, we can use similar arguments as in (i) (but with Lemma 8.10(ii)(c) replaced by Lemma 8.10(ii)(b) in case of Assumption 7.3) to obtain

$$(8.6) \qquad \sup_{|\eta - \eta_{b_n}(t)|_1 < \iota} |\nabla_\eta^2 L_{n,b_n}^{\circ,c}(t, \eta) - V^\circ(t, \eta)| = o_\mathbb{P}(1),$$

where

$$(8.7) \qquad V^\circ(t, \eta) = \int_{-1}^1 K(x) \begin{pmatrix} 1 & x \\ x & x^2 \end{pmatrix} \otimes V(t, \eta_1 + \eta_2 x) dx.$$

Let

$$(8.8) \qquad V^\circ(t) := \begin{pmatrix} 1 & 0 \\ 0 & \mu_{K,2} \end{pmatrix} \otimes V(t).$$

From (i), we have $|\bar{\eta}(t) - \eta_{b_n}(t)| \leq |\hat{\eta}_{b_n}(t) - \eta_{b_n}(t)| = o_\mathbb{P}(1)$, i.e. $\bar{\eta}_1(t) = \theta(t) + o_\mathbb{P}(1)$ and $\bar{\eta}_2(t) = b_n \theta'(t) + o_\mathbb{P}(1) = o_\mathbb{P}(1)$. By continuity of $\theta \mapsto V(t, \theta)$ and (8.6), we conclude that

$$(8.9) \qquad \nabla_\theta^2 L_{n,b_n}^{\circ,c}(t, \bar{\eta}(t)) = V^\circ(t, \bar{\eta}(t)) + o_\mathbb{P}(1) = V^\circ(t) + o_\mathbb{P}(1).$$

By Lemma 8.10(i), we have

$$(8.10) \qquad \|\nabla_\eta L_{n,b_n}^{\circ,c}(t, \eta_{b_n}(t)) - \nabla_\eta \hat{L}_{n,b_n}^\circ(t, \eta_{b_n}(t))\|_1 = O((nb_n)^{-1}).$$

With (8.5), (8.9) and (8.10) we obtain

$$\begin{pmatrix} \sqrt{nb_n}(\hat{\theta}_{b_n}(t) - \theta(t)) \\ \sqrt{nb_n^3}(\hat{\theta}_{b_n}'(t) - \theta'(t)) \end{pmatrix} = \sqrt{nb_n}(\hat{\eta}_{b_n}(t) - \eta_{b_n}(t))$$

$$= -V^\circ(t)^{-1}\sqrt{nb_n}\nabla_\eta \hat{L}_{n,b_n}^\circ(t, \eta_{b_n}(t)) + o_\mathbb{P}(1)$$

$$= -V^\circ(t)^{-1}\sqrt{nb_n}\{\nabla_\eta \hat{L}_{n,b_n}^\circ(t, \eta_{b_n}(t)) - \mathbb{E}\nabla_\eta \hat{L}_{n,b_n}^\circ(t, \eta_{b_n}(t))\}$$

$$(8.11) \qquad\qquad -V^\circ(t)^{-1} \begin{pmatrix} \sqrt{nb_n}\mathbb{E}\nabla_{\eta_1} \hat{L}_{n,b_n}^\circ(t, \eta_{b_n}(t)) \\ \sqrt{nb_n^3}b_n^{-1}\mathbb{E}\nabla_{\eta_2} \hat{L}_{n,b_n}^\circ(t, \eta_{b_n}(t)) \end{pmatrix} + o_\mathbb{P}(1).$$

By (8.11), it is enough to show the two convergences in probability,

$$(8.12) \qquad b_n^{-1}\{\nabla_{\eta_2} L_{n,b_n}^\circ(t, \eta_{b_n}(t)) - \mathbb{E}\nabla_\eta \hat{L}_{n,b_n}^\circ(t, \eta_{b_n}(t))\} = o_\mathbb{P}(1),$$

$$(8.13) \qquad b_n^{-1}\mathbb{E}\nabla_\eta \hat{L}_{n,b_n}^\circ(t, \eta_{b_n}(t)) = o_\mathbb{P}(1).$$

By (8.56) (use Lemma 8.7(i) if Assumption 7.1 holds and Lemma 8.8(i) if Assumption 7.3 holds) from the proof of Lemma 8.10(ii), applied to each component of $\nabla_\theta \ell$ with $\hat{K}(x) = K(x)x$ and $\varsigma = 1$, we obtain

$$\|\nabla_{\eta_2} L_{n,b_n}^\circ(t, \eta_{b_n}(t)) - \mathbb{E}\nabla_\eta \hat{L}_{n,b_n}^\circ(t, \eta_{b_n}(t))\|_2 = O((nb_n)^{-1/2}),$$

which shows (8.12) due to $nb_n^3 \to \infty$. Using the intermediate result (8.63) in the proof of Lemma 8.12, we have

$$\mathbb{E}\nabla_\eta \hat{L}_{n,b_n}^\circ(t, \eta_{b_n}(t)) = O(b_n^3 + n^{-1} + (nb_n)^{-1}),$$

we obtain (8.13) due to $nb_n^3 \to \infty$, which completes the proof of (i).

(ii),(iii) Our aim is to show asymptotic normality of the term in the second to last line of (8.11). Define $U_{i,n}(t) := (K_{b_n}(i/n - t), K_{b_n}(i/n - t)(i/n - t)b_n^{-1})^\mathsf{T}$. Following the proof idea of Theorem 3(ii) in [50], let $m \geq 1$ and define

$$S_{n,b_n,m}(t) := \sum_{l=0}^{m-1} (nb_n)^{-1/2} \sum_{i=1}^n U_{i,n}(t) \otimes P_{i-l}\nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(t) + \theta'(t)(i/n - t)).$$

Recall $\eta_{b_n}(t) = (\theta(t)^\mathsf{T}, b_n\theta'(t)^\mathsf{T})^\mathsf{T}$. Write shortly LIM for $\limsup_{n\to\infty}\limsup_{m\to\infty}$. Then we have for each component $j = 1, \ldots, 2d_\Theta$, that

$$\text{LIM } \|S_{n,b_n,m}(t)_j - (nb_n)^{1/2}\{\nabla_{\eta_j}\hat{L}_{n,b_n}^\circ(t, \eta_{b_n}(t)) - \mathbb{E}\nabla_{\eta_j}\hat{L}_{n,b_n}^\circ(t, \eta_{b_n}(t))\}\|_2$$

$$\leq \text{ LIM } (nb_n)^{-1/2} \sum_{l=m}^\infty \Big\| \sum_{i=1}^n (U_{i,n}(t) \otimes P_{i-l}\nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(t) + \theta'(t)(i/n - t)))_j \Big\|_2$$

$$= \text{ LIM } (nb_n)^{-1/2} \Big(\sum_{i=1}^n \big\|(U_{i,n}(t) \otimes P_{i-l}\nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(t) + \theta'(t)(i/n - t)))_j\big\|_2^2 \Big)^{1/2}$$

$$(8.14)\leq |K|_\infty \text{LIM} \sum_{l=m}^\infty \sup_{t\in[0,1]} \sup_{i,\theta} \delta_2^{\nabla_{\theta_i}\ell(\tilde{Z}(t),\theta)}(l) = 0,$$

by Lemma 8.7(i) if Assumption 7.1 holds or Lemma 8.8(i) if Assumption 7.3 holds. Define $M_i(t) := (nb_n)^{-1/2}\sum_{l=0}^{m-1} U_{i,n}(t) \otimes P_i\nabla_\theta \ell(\tilde{Z}_{i+l}((i+l)/n), \theta(t) + \theta'(t)(i/n - t))$ and $\tilde{S}_{n,b_n,m}(t) := \sum_{i=1}^n M_i(t)$. It is easy to see with Lemma 8.3 or Lemma 8.5 applied to $\nabla_\theta \ell$ that with some $\iota' > 0$ small enough,

$$(8.15) \qquad \sup_{|u-v|\leq \iota'} \|P_i\nabla_\theta \ell(\tilde{Z}_0(u), \theta(v))\|_2 \leq 2 \sup_{|u-v|\leq \iota'} \|\nabla_\theta \ell(\tilde{Z}_0(u), \theta(v))\|_2 < \infty.$$

Since m is finite and (8.15), we conclude that for each component $j = 1, \ldots, 2d_\Theta$,

$$(8.16) \qquad \|S_{n,b_n,m}(t)_j - \tilde{S}_{n,b_n,m}(t)_j\|_2 = O((nb_n)^{-1/2}).$$

Let $a = (a_1^\mathsf{T}, a_2^\mathsf{T})^\mathsf{T} \in \mathbb{R}^{d_\Theta} \times \mathbb{R}^{d_\Theta}$. We want to apply a central limit theorem for martingale differences to $a^\mathsf{T}\tilde{S}_{n,b_n,m}(t)$. Put

$$\Sigma_m := \sum_{l_1,l_2=0}^{m-1} \mathbb{E}\big[P_0\nabla_\theta \ell(\tilde{Z}_{l_1}(t), \theta(t))P_0\nabla_\theta \ell(\tilde{Z}_{l_2}(t), \theta(t))^\mathsf{T}\big] = \text{Cov}\Big(\sum_{l=0}^{m-1} P_0\nabla_\theta \ell(\tilde{Z}_l(t), \theta(t))\Big).$$

Lemma 8.3 (if Assumption 7.1 holds) or Lemma 8.5 (if Assumption 7.3 holds; note that $|i/n - t| \leq b_n$ implies both $\theta(t) + \theta'(t)(i/n - t)$ and $\theta(\frac{i+l_1}{n})$ to be near $\theta(t)$) applied to $\nabla_\theta \ell$ gives:

$$\sup_{|i/n-t| \leq b_n} \|P_i \nabla_\theta \ell(\tilde{Z}_{i+l_1}((i+l_1)/n), \theta(t) + \theta'(t)(i/n - t)) - P_i \nabla_\theta \ell(\tilde{Z}_{i+l_1}(t), \theta(t))\|_2$$

$$\leq \sup_{|i/n-t| \leq b_n} \|\nabla_\theta \ell(\tilde{Z}_0((i+l_1)/n), \theta(t) + \theta'(t)(i/n - t)) - \nabla_\theta \ell(\tilde{Z}_0(t), \theta(t))\|_2$$

$$= O(b_n + n^{-1})$$

and due to (8.15),

$$\sup_i \|P_i \nabla_\theta \ell(\tilde{Z}_{i+l_2}((i+l_2)/n), \theta(t) + \theta'(t)(i/n - t))\|_2 \leq 2 \sup_{|u-v| \leq \iota'} \|\nabla_\theta \ell(\tilde{Z}_0(u), \theta(v))\|_2 < \infty.$$

We therefore have by Hölder's and Markov's inequality that

$$\sum_{i=1}^n M_i(t) M_i(t)^\mathsf{T}$$

$$= \sum_{l_1,l_2=0}^{m-1} (nb_n)^{-1} \sum_{i=1}^n K_{b_n}(i/n - t)^2 \begin{pmatrix} 1 & (i/n - t)b_n^{-1} \\ (i/n - t)b_n^{-1} & (i/n - t)^2 b_n^{-2} \end{pmatrix}$$

$$\otimes \{P_i \nabla_\theta \ell(\tilde{Z}_{i+l_1}(t), \theta(t)) \cdot P_i \nabla_\theta \ell(\tilde{Z}_{i+l_2}(t), \theta(t))^\mathsf{T}\} + O_\mathbb{P}(b_n + n^{-1})$$

$$= \begin{pmatrix} \sigma_{K,0}^2 & 0 \\ 0 & \sigma_{K,2}^2 \end{pmatrix} \otimes \Sigma_m + o_\mathbb{P}(1).$$

The last step is due to Lemma A.2 in [14]. It remains to show a Lindeberg-type condition for $M_i(t)$. Put $\tilde{M}_{ij,l} := P_i \nabla_{\theta_j} \ell(\tilde{Z}_{i+l}((i+l)/n), \theta(t) + \theta'(t)(i/n - t))$. There exists some constant $C > 0$ such that for $j = 1, \ldots, d_\Theta$ and $\iota > 0$,

$$\sum_{i=1}^n \mathbb{E}\big[M_i(t)_j^2 \mathbf{1}_{\{|M_i(t)_j| > \iota\}}\big]$$

(8.17)
$$\leq C(nb_n)^{-1} \sum_{l=0}^{m-1} \sum_{i=1}^n K_{b_n}(i/n - t)^2 \mathbb{E}\big[\tilde{M}_{ij,l}^2 \mathbf{1}_{\{|K|_\infty |\tilde{M}_{ij,l}| > \iota(nb_n)^{1/2}\}}\big].$$

Using Hölder's inequality we have

$$\mathbb{E}\big[\tilde{M}_{ij,l}^2 \mathbf{1}_{\{|K|_\infty |\tilde{M}_{ij,l}| > \iota(nb_n)^{1/2}\}}\big] \leq \mathbb{E}[|\tilde{M}_{ij,l}|^{2+a}]^{2/(2+a)} \mathbb{P}(|K|_\infty |\tilde{M}_{ij,l}| > \iota(nb_n)^{1/2})^{a/(2+a)},$$

which tends to zero using Markov's inequality, (8.15) (with $\|\cdot\|_2$ replaced by $\|\cdot\|_{2+a}$) and the condition

$$\sup_{j \in \mathbb{N}_0} \sup_{t \in [0,1]} \|\tilde{Z}_0(t)_j\|_{(2+a)M} < \infty$$

(which is automatically fulfilled with $a = \varsigma$ if Assumption 7.3 holds with $r = 2 + \varsigma$). This shows that (8.17) is tending to 0. The proof for $j = d_\Theta + 1, \ldots, 2d_\Theta$ is similar. From Theorem 18.2 in Billingsley [4] and the Cramer-Wold device we obtain

$$(8.18) \qquad \tilde{S}_{n,b_n,m}(t) \xrightarrow{d} N\left(0, \begin{pmatrix} \sigma_{K,0}^2 & 0 \\ 0 & \sigma_{K,2}^2 \end{pmatrix} \otimes \Sigma_m\right).$$

Using Theorem 5.46 in [49], (8.14), (8.16), (8.18) and $\Sigma_m \to \Lambda(t)$ $(m \to \infty)$, we obtain

$$(8.19) \quad (nb_n)^{1/2}\{\nabla_\eta \hat{L}_{n,b_n}^\circ(t, \eta_{b_n}(t)) - \mathbb{E}\nabla_\eta \hat{L}_{n,b_n}^\circ(t, \eta_{b_n}(t))\} \xrightarrow{d} N\left(0, \begin{pmatrix} \sigma_{K,0}^2 & 0 \\ 0 & \sigma_{K,2}^2 \end{pmatrix} \otimes \Lambda(t)\right).$$

Using (8.19), the expansion (8.11) and Lemma 8.12, we obtain the result provided that $nb_n^7 \to 0$ for (ii) and $nb_n^9 \to 0$ for (iii). $\qquad \square$

PROOF OF THEOREM 3.2. By Proposition 9.1, Assumptions 7.1 and 7.2 are fulfilled. By Proposition 9.2, Assumptions 7.3 and 7.4 are fulfilled. The results now follow from Theorem 8.2. $\qquad \square$

THEOREM 8.2 (Weak Bahadur representation of $\hat{\theta}_{b_n}, \widehat{\theta}'_{b_n}$). *Let* $\beta_n = (nb_n)^{-1/2}b_n^{-1/2}\log(n)^{1/2}$ *and put*

$$\tau_n^{(j)} = (\beta_n + b_n)((nb_n)^{-1/2}\log(n) + b_n^{1+j}), \qquad j = 1, 2.$$

Let Assumption 7.1 be fulfilled with some $r > 2$.

(i) *It holds that*

$$(8.20) \quad \sup_{t \in \mathcal{T}_n} \left| V(t) \cdot \{\hat{\theta}_{b_n}(t) - \theta(t)\} - \nabla_\theta L_{n,b_n}^c(t, \theta(t), \theta'(t)) \right| = O_\mathbb{P}(\tau_n^{(1)}),$$

$$(8.21) \quad \sup_{t \in \mathcal{T}_n} \left| \nabla_\theta L_{n,b_n}^c(t, \theta(t), \theta'(t)) - b_n^2 \frac{\mu_{K,2}}{2} V(t)\theta''(t) \right.$$

$$\left. -(nb_n)^{-1}\sum_{i=1}^n K_{b_n}(i/n - t)h_i \right| = O_\mathbb{P}(\beta_n b_n^2 + b_n^3 + (nb_n)^{-1}).$$

(ii) *If additionally Assumption 7.2 is fulfilled, then*

$$(8.22) \quad \sup_{t \in \mathcal{T}_n} \left| \mu_{K,2} V(t) \cdot b_n\{\widehat{\theta}'_{b_n}(t) - \theta'(t)\} - b_n^{-1}\nabla_{\theta'} L_{n,b_n}^c(t, \theta(t), \theta'(t)) \right| = O_\mathbb{P}(\tau_n^{(2)}),$$

$$(8.23) \quad \sup_{t \in \mathcal{T}_n} \left| b_n^{-1}\nabla_{\theta'} L_{n,b_n}^c(t, \theta(t), \theta'(t)) - b_n^3 \frac{\mu_{K,4}}{2} V(t)bias(t) \right.$$

$$\left. -(nb_n)^{-1}\sum_{i=1}^n K_{b_n}(i/n - t)\frac{(i/n - t)}{b_n}h_i \right| = O_\mathbb{P}(\beta_n b_n^2 + b_n^4 + (nb_n)^{-1}).$$

PROOF OF THEOREM 8.2. (i),(ii) By Lemma 8.10(i),(iii)(a) and Lemma 8.11(a) (in case Assumption 7.1 holds) or Lemma 8.10(i),(iii)(c) and Lemma 8.11(a) (if Assumption 7.3 holds) applied to $g = \ell$, we have that

$$\sup_{t \in \mathcal{T}_n} \sup_{\eta \in E_n} |L_{n,b_n}^{\circ,c}(t,\eta) - L^{\circ}(t,\eta)| = O_{\mathbb{P}}(\beta_n + (nb_n)^{-1}) + O(b_n),$$

i.e. $L_{n,b_n}^{\circ,c}(t,\eta)$ converges to $L^{\circ}(t,\eta)$ uniformly in t,η if $b_n = o(1)$ and $\beta_n = o(1)$. It was already seen in the proof of Theorem 8.1 that $L^{\circ}(t,\eta)$ is continuous w.r.t. η and uniquely minimized by $\eta = (\theta(t)^{\mathsf{T}}, 0)^{\mathsf{T}}$. Standard arguments give

$$(8.24) \qquad \sup_{t \in \mathcal{T}_n} |\hat{\eta}_{b_n}(t) - \eta_{b_n}(t)| = o_{\mathbb{P}}(1).$$

Thus for n large enough, $\hat{\eta}_{b_n}(t)$ is in the interior of E_n uniformly in t. By a Taylor expansion, we obtain for each $t \in \mathcal{T}_n$:

$$(8.25) \qquad \hat{\eta}_{b_n}(t) - \eta_{b_n}(t) = -\left[V^{\circ}(t) + R_{n,b_n}(t)\right]^{-1} \cdot \nabla_{\eta} L_{n,b_n}^{\circ,c}(t, \eta_{b_n}(t)),$$

where $R_{n,b_n}(t) = \nabla_{\eta}^2 L_{n,b_n}^{\circ,c}(t,\bar{\eta}(t)) - V^{\circ}(t)$ with some $\bar{\eta}(t) \in E_n$ satisfying $|\bar{\eta}(t) - \eta_{b_n}(t)|_1 \leq |\hat{\eta}_{b_n}(t) - \eta_{b_n}(t)|_1$ and $V^{\circ}(t)$ is defined in (8.8).

By Lemma 8.10(i),(iii)(a) and Lemma 8.11(a) (if Assumption 7.1 holds) or Lemma 8.10(i),(iii)(c) and Lemma 8.11(b) (if Assumption 7.3 holds) applied to $g = \nabla_{\theta}^2 \ell$ and $\hat{K}(x) = K(x)$, $\hat{K}(x) = K(x)x$ or $\hat{K}(x) = K(x)x^2$, respectively, we have for some fixed $\iota' > 0$:

$$(8.26) \qquad \sup_{t \in \mathcal{T}_n} \sup_{|\eta - \eta_{b_n}(t)| < \iota'} |\nabla_{\eta}^2 L_{n,b_n}^{\circ,c}(t,\eta) - V^{\circ}(t,\eta)| = O_{\mathbb{P}}(\beta_n + (nb_n)^{-1}) + O(b_n),$$

where $V^{\circ}(t,\eta)$ is defined in (8.7).

For the moment, let $\tilde{h}_i(t) = \nabla_{\theta}\ell(\tilde{Z}_i(t),\theta(t))$. Note that $\mathbb{E}\tilde{h}_0(t) = \mathbb{E}\nabla_{\theta}\ell(\tilde{Z}_0(t),\theta(t)) = 0$ by Assumption 7.1(A3), (A1) (or Assumption 7.3(A3'), (A1')).

By Lemma 8.7(i) (if Assumption 7.1 holds) or Lemma 8.8(i) (if Assumption 7.3 holds), we have $\sup_t \delta_{2+\varsigma}^{\tilde{h}(t)_j}(k) = \sup_t \delta_{2+\varsigma}^{\nabla_{\theta_j}\ell(\tilde{Z}(t),\theta(t))}(k) = O(k^{-(1+\gamma)})$ for each $j = 1,\ldots,d_{\Theta}$. Using Lemma 8.3 (if Assumption 7.1 holds) or Lemma 8.5 (if Assumption 7.3 holds), we see that the assumptions of Lemma 8.16 are fulfilled and thus, applied to $\tilde{h}_i(t)$,

$$(8.27) \qquad \sup_{t \in \mathcal{T}_n} \left|(nb_n)^{-1}\sum_{i=1}^{n} K_{b_n}(i/n - t)\nabla_{\theta}\ell(\tilde{Z}_i(i/n),\theta(i/n))\right| = O_{\mathbb{P}}((nb_n)^{-1/2}\log(n)).$$

With Lemma 8.14, we obtain

$$\sup_{t \in \mathcal{T}_n} \left|\nabla_{\eta}\hat{L}_{n,b_n}^{\circ}(t,\eta_{b_n}(t)) - \mathbb{E}\nabla_{\eta}\hat{L}_{n,b_n}^{\circ}(t,\eta_{b_n}(t))\right| = O_{\mathbb{P}}((nb_n)^{-1/2}\log(n) + \beta_n b_n^2).$$

Since $\mathbb{E}\nabla_\theta \ell(\tilde{Z}_0(t), \theta(t)) = 0$, we obtain with Lemma 8.12(i),(ii) and Lemma 8.10(i):

$$(8.28) \qquad \sup_{t\in\mathcal{T}_n} |\nabla_{\eta_j} L_{n,b_n}^{\circ,c}(t, \eta_{b_n}(t))| = O_{\mathbb{P}}((nb_n)^{-1/2}\log(n) + (nb_n)^{-1} + \beta_n b_n^2 + b_n^{1+j}),$$

where $j = 1, 2$. Since $\theta \mapsto V(t, \theta) = \mathbb{E}\nabla_\theta^2 \ell(\tilde{Z}_0(t), \theta)$ is Lipschitz continuous (apply Lemma 8.3 in case of Assumption 7.1 or Lemma 8.5 in case of Assumption 7.3 to $\nabla^2 \ell$), the same holds for $\eta \mapsto V^\circ(t, \eta)$. We conclude that with some constant $C > 0$,

$$(8.29) \qquad \sup_{t\in\mathcal{T}_n} |R_{n,b_n}(t)| \le \sup_{t\in\mathcal{T}_n} \sup_{\eta\in E_n} |\nabla_\eta^2 L_{n,b_n}^{\circ,c}(t, \eta) - V^\circ(t, \eta)| + C \sup_{t\in\mathcal{T}_n} |\hat{\eta}_{b_n}(t) - \eta_{b_n}(t)|.$$

Inserting (8.28), (8.29) and (8.24) into (8.25), we obtain

$$(8.30) \qquad \sup_{t\in\mathcal{T}_n} |\hat{\eta}_{b_n,j}(t) - \eta_{b_n,j}(t)| = O_{\mathbb{P}}((nb_n)^{-1/2}\log(n) + (nb_n)^{-1} + \beta_n b_n^2 + b_n^{1+j}),$$

where $j = 1, 2$. Inserting (8.30), (8.26) into (8.29), we get $\sup_{t\in\mathcal{T}_n} |R_{n,b_n}(t)| = O_{\mathbb{P}}(\beta_n + b_n + (nb_n)^{-1})$. Together with

$$\left|V^\circ(t)(\hat{\eta}_{b_n}(t) - \eta_{b_n}(t)) - \nabla L_{n,b_n}^{\circ,c}(t, \eta_{b_n}(t))\right|$$
$$\le \left|[I_{2k\times 2k} + V^\circ(t)^{-1}R_{n,b_n}(t)]^{-1} - I_{2k\times 2k}^{-1}\right| \cdot |\nabla_\eta L_{n,b_n}^{\circ,c}(t, \eta_n(t))|$$
$$\le \left|[I_{2k\times 2k} + V^\circ(t)^{-1}R_{n,b_n}(t)]^{-1}\right| \cdot \left|V^\circ(t)^{-1}R_{n,b_n}(t)\right| \cdot |\nabla_\eta L_{n,b_n}^{\circ,c}(t, \eta_{b_n}(t))|,$$

and (8.28) we have (8.20) and (8.22). The other results (8.21) and (8.23) follow from Lemma 8.10(i), Lemma 8.14 and Lemma 8.12. $\qquad\qquad\square$

LEMMA 8.3. *Let $q > 0$ and $s \ge 0$. Let $g \in \mathcal{H}_s(M_y, M_x, \chi, \bar{C})$ and $M := \max\{M_x, M_y\}$. Let $\hat{Y}, \hat{Y}'$ be random variables and $\hat{X} = (\hat{X}_j)_{j\in\mathbb{N}}$, $\hat{X}' = (\hat{X}_j')_{j\in\mathbb{N}}$ be sequences of random variables. Assume that there exists some $D > 0$ such that uniformly in $j \in \mathbb{N}$,*

$$(8.31) \qquad \|\hat{Y}\|_{qM(1+s)}, \qquad \|\hat{Y}'\|_{qM(1+s)}, \qquad \|\hat{X}_j\|_{qM(1+s)}, \qquad \|\hat{X}_j'\|_{qM(1+s)} \le D.$$

Let $\hat{Z} = (\hat{Y}, \hat{X})$, $\hat{Z}' = (\hat{Y}', \hat{X}')$. Then there exists some constant $C > 0$ only dependent on M, D, χ and $\tilde{D}$ (only in (ii)) such that

(i)

$$(8.32) \qquad \|\sup_{\theta\in\Theta} |g(\hat{Z}, \theta) - g(\hat{Z}', \theta)|\|_q \le \bar{C} \cdot C \sum_{j=0}^\infty \hat{\chi}_j \|\hat{Z}_j - \hat{Z}_j'\|_{qM},$$

$$(8.33) \qquad \left\|\sup_{\theta\neq\theta'} \frac{|g(\hat{Z}, \theta) - g(\hat{Z}, \theta')|}{|\theta - \theta'|_1}\right\|_q \le \bar{C} \cdot C,$$

$$(8.34) \qquad \|\sup_{\theta\in\Theta} |g(\hat{Z}, \theta)|\|_q \le \bar{C} \cdot C,$$

$$(8.35) \qquad \|R_{M_x,M_y}(\hat{Z})^{1+s}\|_q \le C.$$

(ii) *Let* $s = 0$. *If additionally,* $\mathbb{E}[|\hat{Y} - \hat{Y}'|^{qM_y}|\sigma(\hat{X}, \hat{X}')] \leq \tilde{D}|\hat{X} - \hat{X}'|_{\chi,1}^{qM_y}$ *with some constant* $\tilde{D} > 0$, *then*

$$(8.36) \qquad \|\sup_{\theta \in \Theta} |g(\hat{Z}, \theta) - g(\hat{Z}', \theta)|\|_q \leq \bar{C} \cdot C \sum_{j=1}^{\infty} \chi_j \|\hat{X}_j - \hat{X}_j'\|_{qM}.$$

PROOF OF LEMMA 8.3. During the proofs, we consider $M_y, M_x \geq 2$ and thus $M \geq 2$. In the case $M_y = 1$ or $M_x = 1$, the proofs are easier since some terms do not show up.

(i) Note that R_{M_y-1,M_x-1} is a polynomial in $|x|_{\chi,1}, |y|$ with (joint) degree at most $M-1$. Since

$$R_{M_y-1,M_x-1}(\hat{Z}) = \sum_{k+l \leq M-1, 0 \leq k \leq M_y-1, 0 \leq l \leq M_x-1} |\hat{Y}|^k |\hat{X}|_{\chi,1}^l,$$

we have by Hölder's inequality,

$$\|R_{M_y-1,M_x-1}(\hat{Z})\|_{q(1+s)M/(M-1)}$$

$$\leq \sum_{k+l \leq M-1, 0 \leq k \leq M_y-1, 0 \leq l \leq M_x-1} \left(\sum_{i=1}^{\infty} \chi_i \|\hat{X}_i\|_{q(1+s)M}\right)^l \|\hat{Y}\|_{q(1+s)M}^k$$

$$(8.37) \qquad \leq \sum_{0 \leq k+l \leq M-1} (|\chi|_1 D)^l D^k \leq (1 + D(|\chi|_1 + 1))^{M-1}.$$

Therefore:

$$\|\sup_{\theta \in \Theta} |g(\hat{Z}, \theta) - g(\hat{Z}', \theta)|\|_q$$

$$\leq \bar{C}\big\||\hat{Y} - \hat{Y}'|(R_{M_y-1,M_x-1}(\hat{Y}, \hat{X})^{1+s} + R_{M_y-1,M_x-1}(\hat{Y}', \hat{X})^{1+s})\big\|_q$$

$$(8.38) \qquad + \big\||\hat{X} - \hat{X}'|_{\chi,1} \cdot (R_{M_y-1,M_x-1}(\hat{Y}, \hat{X})^{1+s} + R_{M_y-1,M_x-1}(\hat{Y}, \hat{X}')^{1+s})\big\|_q$$

$$\leq \bar{C}\|\hat{Y} - \hat{Y}'\|_{qM} \cdot \big(\|R_{M_y-1,M_x-1}(\hat{Y}, \hat{X})\|_{q(1+s)M/(M-1)}^{1+s}$$

$$+ \|R_{M_y-1,M_x-1}(\hat{Y}', \hat{X})\|_{q(1+s)M/(M-1)}^{1+s}\big)$$

$$+ \bar{C}\||\hat{X} - \hat{X}'|_{\chi,1}\|_{qM} \big(\|R_{M_y-1,M_x-1}(\hat{Y}, \hat{X})\|_{q(1+s)M/(M-1)}^{1+s}$$

$$+ \|R_{M_y-1,M_x-1}(\hat{Y}, \hat{X}')\|_{q(1+s)M/(M-1)}^{1+s}\big)$$

$$\leq 2\bar{C}(1 + D(|\chi|_1 + 1))^{(M-1)(1+s)} \big(\|\hat{Y} - \hat{Y}'\|_{qM} + \sum_{j=1}^{\infty} \chi_j \|\hat{X}_j - \hat{X}_j'\|_{qM}\big),$$

which shows (8.32). The proof of (8.34) is obvious from (8.32) and $\sup_{\theta \in \Theta} |g(0, \theta)| \leq \bar{C}$.

R_{M_y,M_x} is a polynomial in $|x|_{\chi,1}$ and $|y|$ with (joint) degree at most M. As in (8.37), we obtain

$$\|R_{M_y,M_x}(\hat{Z})\|_{q(1+s)} \leq (1 + D(|\chi|_1 + 1))^M,$$

showing (8.35).

(8.33) follows from (8.35) and

$$|g(\hat{Z}, \theta) - g(\hat{Z}, \theta')| \leq \bar{C}|\theta - \theta'|_1 R_{M_y, M_x}(\hat{Z})^{1+s}.$$

(ii) We first obtain (8.38) as before. The second summand has the upper bound

$$2\bar{C}(1 + D(|\chi|_1 + 1))^{M-1} \sum_{j=1}^{\infty} \chi_j \|\hat{X}_j - \hat{X}_j'\|_{qM}.$$

For the first summand in (8.38), notice that

$$\left\| |\hat{Y} - \hat{Y}'| \cdot R_{M_y-1, M_x-1}(\hat{Y}, \hat{X}) \right\|_q$$

$$\leq \sum_{k+l \leq M-1, 0 \leq k \leq M_y-1, 0 \leq l \leq M_x-1} \left\| |\hat{Y} - \hat{Y}'| \cdot |\hat{Y}|^k \cdot |\hat{X}|_{\chi,1}^l \right\|_q.$$

By Hölder's inequality for conditional expectations,

$$\left\| |\hat{Y} - \hat{Y}'| \cdot |\hat{Y}|^k |\hat{X}|_{\chi,1}^l \right\|_q$$

$$= \mathbb{E}\big[\mathbb{E}[|\hat{Y} - \hat{Y}'|^q|\hat{Y}|^{qk}|\sigma(\hat{X}, \hat{X}')] \cdot |\hat{X}|_{\chi,1}^{ql}\big]^{1/q}$$

$$\leq \mathbb{E}\big[\mathbb{E}[|\hat{Y} - \hat{Y}'|^{qM_y}|\sigma(\hat{X}, \hat{X}')]^{1/M_y}\mathbb{E}[|\hat{Y}|^{qkM_y/(M_y-1)}|\sigma(\hat{X}, \hat{X}')]^{(M_y-1)/M_y}|\hat{X}|_{\chi,1}^{ql}\big]^{1/q}$$

$$=: \mathbb{E}[A_1 \cdot A_2 \cdot A_3]^{1/q}.$$

By the additional condition, we have $A_1 \leq \tilde{D}^{1/M_y}|\hat{X} - \hat{X}'|_{\chi,1}^q$. By Hölder's inequality,

$$\mathbb{E}[A_1 \cdot A_2 \cdot A_3]^{1/q} \leq \mathbb{E}[A_1^M]^{1/(qM)}\mathbb{E}[A_2^{M/k}]^{k/(qM)}\mathbb{E}[A_3^{M/(M-k-1)}]^{(M-k-1)/(qM)}.$$

We have $\mathbb{E}[A_1^M]^{1/M} \leq \tilde{D}^{1/M_y}\||\hat{X} - \hat{X}'|_{\chi,1}\|_{qM}^q$,

$$\mathbb{E}[A_3^{M/(M-k-1)}]^{(M-k-1)/M} = \||\hat{X}|_{\chi,1}\|_{Mql/(M-k-1)}^{ql} \leq \||\hat{X}|_{\chi,1}\|_{qM}^{ql}$$

and by Jensen's inequality for conditional expectations (note that $\frac{M_y-1}{k}\frac{M}{M_y} \geq 1$),

$$\mathbb{E}[A_2^{M/k}]^{k/M} \leq \mathbb{E}[\mathbb{E}[|\hat{Y}|^{qkM_y/(M_y-1)\cdot\frac{M_y-1}{M_y}\frac{M}{k}}|\sigma(\hat{X}, \hat{X}')]]^{k/M} = \|\hat{Y}\|_{Mq}^{qk}.$$

Putting the results together we obtain

$$\left\| |\hat{Y} - \hat{Y}'| \cdot |\hat{Y}|^k |\hat{X}|_{\chi,1}^l \right\|_q \leq \tilde{D}^{1/(qM_y)} \sum_{i=1}^{\infty} \chi_i \|\hat{X}_i - \hat{X}_i'\|_{qM} \cdot D^k(|\chi|_1 D)^l,$$

which leads to

$$\left\| |\hat{Y} - \hat{Y}'| \cdot R_{M_y-1, M_x-1}(\hat{Y}, \hat{X}) \right\|_q \leq \tilde{D}^{1/(qM_y)}(1 + D(1 + |\chi|_1))^{M-1} \cdot \sum_{i=1}^{\infty} \chi_i \|\hat{X}_i - \hat{X}_i'\|_{qM},$$

giving the result.   □

LEMMA 8.4 (for tvGARCH). *Let $q > 0$ and $s > 0$. Let $\hat{X}, \hat{X}', \hat{Y}, \hat{Y}'$ be as in Lemma 8.3 satisfying (8.31). Let $g = \ell$ satisfy (7.6). Then there exists some constant $C^{(s)} > 0$ only dependent on M, D, $\chi^{(s)}$ such that*

$$(8.39)\quad \|\sup_{\theta \in \Theta} |g(\hat{Z}, \theta) - g(\hat{Z}', \theta)|\|_q \;\leq\; \bar{C}^{(s)} \cdot C^{(s)} \sum_{j=0}^{\infty} \hat{\chi}_j^{(s)} \big(\|\hat{Z}_j - \hat{Z}'_j\|_{qM} + \|\hat{Z}_j - \hat{Z}'_j\|_{qM(1+s)}^s\big),$$

$$(8.40)\qquad \|\sup_{\theta \in \Theta} |g(\hat{Z}, \theta)|\|_q \;\leq\; \bar{C} \cdot C.$$

PROOF OF LEMMA 8.4. It holds that

$$\big\|\sup_{\theta \in \Theta} |g(\hat{Z}, \theta) - g(\hat{Z}', \theta)|\big\|_q$$
$$\leq\; \bar{C}^{(s)} \big\| |\hat{Z} - \hat{Z}'|_{\chi^{(s)},s} \cdot (R_{M,M}(\hat{Z}) + R_{M,M}(\hat{Z}'))\big\|_q$$
$$+ \bar{C}^{(s)} \big\| |\hat{Z} - \hat{Z}'|_{\chi^{(s)},1} \cdot (R_{M-1,M-1}(\hat{Z})^{1+s} + R_{M-1,M-1}(\hat{Z}')^{1+s})\big\|_q.$$

The second summand can be dealt with as in the proof of Lemma 8.3(i), giving the upper bound

$$2\bar{C}^{(s)}(1 + D(|\chi^{(s)}|_1 + 1))^{(M-1)(1+s)} \big(\|\hat{Y} - \hat{Y}'\|_{qM} + \sum_{j=1}^{\infty} \chi_j^{(s)} \|\hat{X}_j - \hat{X}'_j\|_{qM}\big).$$

For the first summand, we obtain with Hölder's inequality:

$$\big\| |\hat{Z} - \hat{Z}'|_{\hat{\chi}^{(s)},s}^s \cdot (R_{M,M}(\hat{Z}) + R_{M,M}(\hat{Z}'))\big\|_q$$
$$\leq\; \sum_{j=0}^{\infty} \hat{\chi}_j^{(s)} \|\hat{Z}_j - \hat{Z}'_j\|_{q(M+s)}^s \cdot \big(\|R_{M,M}(\hat{Z})\|_{q(M+s)/M} + \|R_{M,M}(\hat{Z}')\|_{q(M+s)/M}\big),$$

giving the result since $M + s \leq M(1 + s)$ and thus $\|R_{M,M}(\hat{Z})\|_{q(M+s)/M} \leq (1 + D(|\chi^{(s)}|_1 + 1))^M$. $\qquad\square$

LEMMA 8.5 (for tvGARCH). *Let $q > 0, s > 0, \iota > 0$. Let $\hat{X}, \hat{X}', \hat{Y}, \hat{Y}'$ be as in Lemma 8.3 satisfying (8.31). Let $g \in \mathcal{H}_{s,\iota}^{mult}(M_y, M_x, \chi, \bar{C})$. Let ζ_0 be independent of $\hat{X}, \hat{X}'$ with $\|\zeta_0\|_{qM} \leq D$. Then there exists some constant $C > 0$ only dependent on $M, D, \chi, \bar{C}$ such that*

$$(8.41)\qquad \|\sup_{|\theta - \tilde{\theta}| < \iota} |\tilde{g}_{\tilde{\theta}}(\zeta_0, \hat{X}, \theta) - \tilde{g}_{\tilde{\theta}}(\zeta_0, \hat{X}', \theta)|\|_q \;\leq\; \bar{C} \cdot C \sum_{j=1}^{\infty} \chi_j \|\hat{X}_j - \hat{X}'_j\|_{qM},$$

$$(8.42)\quad \Big\| \sup_{\theta \neq \theta', |\theta - \tilde{\theta}|_1 < \iota, |\theta' - \tilde{\theta}|_1 < \iota} \frac{|\tilde{g}_{\tilde{\theta}}(\zeta_0, \hat{X}, \theta) - \tilde{g}_{\tilde{\theta}}(\zeta_0, \hat{X}, \theta')|}{|\theta - \theta'|_1} \Big\|_q \;\leq\; \bar{C} \cdot C,$$

$$(8.43)\qquad \|\sup_{|\theta - \tilde{\theta}|_1 < \iota} |\tilde{g}(\zeta_0, \hat{X}, \theta)|\|_q \;\leq\; \bar{C} \cdot C.$$

PROOF OF LEMMA 8.5. As in the proof of Lemma 8.3(i), we have:

$$\|R_{M_y-1,M_x-1}(1,\hat{X})\|_{q(1+s)M/(M-1)} \leq (1 + D(|\chi|_1 + 1))^{M-1}.$$

Thus, with Hölder's inequality,

$$\Big\| \sup_{|\theta-\tilde{\theta}|<\iota} |g(\zeta_0, \hat{X}, \theta) - g(\zeta_0, \hat{X}', \theta)| \Big\|_q$$

$$\leq\quad 2\bar{C}(1 + D(|\chi|_1 + 1))^{(M-1)(1+s)}(1 + \|\zeta_0\|_{qM}) \sum_{j=1}^{\infty} \chi_j \|\hat{X}_j - \hat{X}'_j\|_{qM}.$$

This shows (8.41). (8.43) follows from (8.41) since

$$\Big\| \sup_{|\theta-\tilde{\theta}|<\iota} |\tilde{g}_{\tilde{\theta}}(\zeta_0, 0, \theta)| \Big\|_q \leq \bar{C}(1 + \|\zeta_0\|_{qM}^M) \leq \bar{C}(1 + D^M).$$

$\square$

LEMMA 8.6.   *Assume that $\theta(\cdot) \in C^2[0,1]$. Let $\chi' = (\tilde{\chi}'_i)_{i\in\mathbb{N}}$ be an absolutely summable sequence. Let $g : \mathbb{R} \times \mathbb{R}^{\mathbb{N}} \times \Theta \to \mathbb{R}$ be continuously differentiable. Suppose that either*

(a) *Assumption 7.1(A5) holds with some $r \geq 2$ and Assumption 7.2(B3) holds. Additionally, $g \in \mathcal{H}(M'_y, M'_x, \chi, \bar{C})$, $\nabla_\theta g \in \mathcal{H}(M'_y, M'_x, \chi, \bar{C})$ and for all $l \in \mathbb{N}$, $\partial_{x_l} g \in \mathcal{H}(M'_y - 1, M'_x - 1, \tilde{\chi}, \bar{C}\chi_l)$.*

or

(b) *(for tvGARCH) There exists $s \geq 0$ such that Assumption 7.3(A5') holds with $r \geq 2(1 + s)$ and Assumption 7.4(B3') holds. Additionally, $g \in \mathcal{H}_{s,\iota}^{mult}(M'_y, M'_x, \chi, \bar{C})$, $\nabla_\theta g \in \mathcal{H}_{s,\iota}^{mult}(M'_y, M'_x, \chi, \bar{C})$ and for all $l \in \mathbb{N}$, $\partial_{x_l} g \in \mathcal{H}_{s,\iota}^{mult}(M'_y - 1, M'_x - 1, \tilde{\chi}, \bar{C}\chi_l)$.*

Define $M' := \max\{M'_y, M'_x\}$. Then

(i) $\sup_{t\in[0,1]} \|\partial_t g(\tilde{Z}_0(t), \theta(t))\|_1 < \infty$,
(ii)

$$\sup_{t\neq t'} \frac{\|\partial_t g(\tilde{Z}_0(t), \theta(t)) - \partial_t g(\tilde{Z}_0(t'), \theta(t'))\|_1}{|t - t'|} < \infty.$$

PROOF OF LEMMA 8.6. (i) Note that

$$(8.44)\qquad \partial_t g(\tilde{Z}_0(t), \theta(t)) = \partial_z g(\tilde{Z}_0(t), \theta(t))\partial_t \tilde{Z}_0(t) + \nabla_\theta g(\tilde{Z}_0(t), \theta(t))\theta'(t).$$

By Lemma 8.3 (if Assumption 7.1 holds) or Lemma 8.5 (if Assumption 7.3 holds),

$$\sup_t \|\nabla_\theta g(\tilde{Z}_0(t), \theta(t))\|_1 < \infty.$$

and there exists a constant $C > 0$ such that for each $j \in \mathbb{N}_0$,

$$(8.45) \qquad \|\partial_{z_j} g(\tilde{Z}_0(t), \theta(t))\|_{M'/(M'-1)} \leq C\bar{C}\hat{\chi}_j.$$

It follows that

$$
\begin{aligned}
\|\partial_z g(\tilde{Z}_0(t), \theta(t))\partial_t \tilde{Z}_0(t)\|_1 &\leq \sum_{j=0}^{\infty} \|\partial_{z_j} g(\tilde{Z}_0(t), \theta(t))\|_{M'/(M'-1)} \cdot \|\partial_t \tilde{Z}_{0j}(t)\|_{M'} \\
&\leq CD\bar{C} \sum_{j=0}^{\infty} \hat{\chi}_j < \infty,
\end{aligned}
$$

which shows the assertion.

(ii) Let $t, t' \in [0,1]$. From Lemma 8.3 (if Assumption 7.1 holds) or Lemma 8.5 (if Assumption 7.3 holds) we obtain with some constant $C > 0$, for each $k = 1, ..., d_\Theta$,

$$
\begin{aligned}
\|\nabla_{\theta_k} g(\tilde{Z}_0(t), \theta(t))\|_1 &\leq C, \\
\|\nabla_{\theta_k} g(\tilde{Z}_0(t), \theta(t)) - \nabla_{\theta_k} g(\tilde{Z}_0(t), \theta(t'))\|_1 &\leq C|\theta(t) - \theta(t')|_1.
\end{aligned}
$$

From Lemma 8.3 (if Assumption 7.1 holds) or Lemma 8.5 (if Assumption 7.3 holds) we obtain for each $k = 1, ..., d_\Theta$ (note that $rM \geq M'$ in Assumption 7.2):

$$
\begin{aligned}
&\|\nabla_{\theta_k} g(\tilde{Z}_0(t), \theta(t')) - \nabla_{\theta_k} g(\tilde{Z}_0(t'), \theta(t'))\|_1 \\
&\leq \bar{C}C\big(\|\tilde{Y}_0(t) - \tilde{Y}_0(t')\|_{M'} + \sum_{j=1}^{\infty} \chi_j \|\tilde{X}_0(t) - \tilde{X}_0(t')\|_{M'}\big) \\
(8.46) \qquad &\leq \bar{C}CC_B|t - t'|(1 + |\chi|_1).
\end{aligned}
$$

By Lipschitz continuity of θ, θ', the above results imply that the second summand in (8.44) fulfills the assertion,

$$\sup_{t \neq t'} \frac{\|\nabla_\theta g(\tilde{Z}_0(t), \theta(t))\theta'(t) - \nabla_\theta g(\tilde{Z}_0(t'), \theta(t'))\theta'(t')\|_1}{|t - t'|} < \infty.$$

It remains to show the same for the first summand in (8.44). By (8.45) and $\|\partial_t \tilde{Z}_{0j}(t) - \partial_t \tilde{Z}_{0j}(t')\|_{M'} \leq C_B|t - t'|$ from Assumption 7.2(B3), we have

$$(8.47) \qquad \|\partial_z g(\tilde{Z}_0(t), \theta(t))(\partial_t \tilde{Z}_0(t) - \partial_t \tilde{Z}_0(t'))\|_1 \leq C\bar{C}C_B|\chi|_1|t - t'|.$$

Similar as in (8.46), we see by Lemma 8.3 (if Assumption 7.1 holds) or Lemma 8.5 (if Assumption 7.3 holds) that

$$(8.48) \qquad \|\partial_{z_j} g(\tilde{Z}_0(t), \theta(t)) - \partial_{z_j} g(\tilde{Z}_0(t'), \theta(t))\|_{M'/(M'-1)} \leq \chi_j \bar{C}CC_B(1 + |\chi'|_1)|t - t'|.$$

Finally, by Lemma 8.3 (if Assumption 7.1 holds) or Lemma 8.5 (if Assumption 7.3 holds) and Lipschitz continuity of θ, we have

$$(8.49) \quad \|\partial_{z_j}g(\tilde{Z}_0(t'),\theta(t))-\partial_{z_j}g(\tilde{Z}_0(t'),\theta(t'))\|_{M'/(M'-1)} \leq \chi_j\bar{C}C|\theta(t)-\theta(t')|_1 = O(|t-t'|).$$

By Hölder's inequality, we conclude from (8.48) and (8.49) that

$$\|(\partial_z g(\tilde{Z}_0(t),\theta(t)) - \partial_z g(\tilde{Z}_0(t'),\theta(t')))\partial_t\tilde{Z}_0(t')\|_1 = O(|t-t'|),$$

which together with (8.47), finishes the proof. $\square$

LEMMA 8.7.   Let $q \geq 1$. Suppose that Assumption 7.1(A5), (A7) hold with some $r \geq q$. Let $g \in \mathcal{H}(M_y, M_x, \chi, \bar{C})$, where $\chi_i = O(i^{-(1+\gamma)})$. Then it holds that

(i) $\sup_{t\in[0,1]} \delta_q^{\sup_\theta |g(\tilde{Z}(t),\theta)|}(j) = O(j^{-(1+\gamma)})$.

(ii) For $M_i(t,\eta,u) := \hat{K}_{b_n}(u-t)g(\tilde{Z}_i(u),\eta_1+\eta_2(u-t)b_n^{-1})$, we have

$$\sup_{u\in[0,1]} \sup_{t,\eta} \delta_q^{M(t,\eta,u)}(j) = O(j^{-(1+\gamma)}), \qquad \sup_{u\in[0,1]} \delta_q^{\sup_{t,\eta} |M(t,\eta,u)|}(j) = O(j^{-(1+\gamma)}).$$

(iii) Let $d_u(t) = \theta(u) - \theta(t) - (u-t)\theta'(t)$ and $M_i^{(2)}(t,u) := \hat{K}_{b_n}(u-t)\{\int_0^1 g(\tilde{Z}_i(u),\theta(t)+sd_u(t))ds\} \cdot d_u(t)$. Then it holds for each component that

$$\sup_{u\in[0,1]} \delta_q^{M^{(2)}(t,u)}(j) = O(b_n^2 j^{-(1+\gamma)}), \qquad \sup_{u\in[0,1]} \delta_q^{\sup_t |M^{(2)}(t,u)|}(j) = O(b_n^2 j^{-(1+\gamma)}).$$

(*) If instead Assumption 7.3(A5'), (A7') hold with some $r > q$ and $g = \ell$ fulfills (7.6) for all $s > 0$ small enough,then the statements above remain valid.

PROOF. (i) Let $\tilde{Z}_j(t)^*$ be a coupled version of $\tilde{Z}_j(t)$ where ζ_0 is replaced by ζ_0^*. By Lemma 8.3 we obtain in case (7.4) that with some constant $\tilde{C} > 0$:

$$\begin{aligned}
&\delta_q^{\sup_\theta |g(\tilde{Z}(t),\theta)|}(j) \\
&= \;\|\sup_\theta |g(\tilde{Z}_j(t),\theta)| - \sup_\theta |g(\tilde{Z}_j(t)^*,\theta)|\|_q \\
&\leq \;\|\sup_\theta |g(\tilde{Z}_j(t),\theta) - g(\tilde{Z}_j(t)^*,\theta)|\|_q \\
&\leq \;\tilde{C}\Big(\|\tilde{Y}_j(t) - \tilde{Y}_j(t)^*\|_{qM} + \sum_{i=1}^\infty \chi_i\|\tilde{X}_{j-i+1}(t) - \tilde{X}_{j-i+1}(t)^*\|_{qM}\Big) \\
(8.50) \quad &\leq \;\tilde{C}\Big(\delta_{qM}^{\tilde{Y}(t)}(j) + \sum_{i=1}^\infty \chi_i\delta_{qM}^{\tilde{X}(t)}(j-i+1)\Big),
\end{aligned}$$

and in case (7.5), similarly

$$(8.51) \qquad \delta_q^{g(\tilde{Z}(t),\theta)}(j) \leq \tilde{C} \sum_{i=1}^{\infty} \chi_i \delta_{qM}^{\tilde{X}(t)}(j-i+1).$$

In case (*), let $s > 0$ be such that $q(1+s) < r$. Then we have by Lemma 8.4:

$$\delta_q^{g(\tilde{Z}(t),\theta)}(j) \leq \tilde{C} \sum_{i=0}^{\infty} \hat{\chi}_i^{(s)} \left(\|\tilde{Z}_i(t) - \tilde{Z}_i(t)^*\|_{qM} + \|\tilde{Z}_i(t) - \tilde{Z}_i(t)^*\|_{qM(1+s)}^s \right)$$

$$(8.52) \qquad \leq \tilde{C} \sum_{i=0}^{\infty} \chi_i^{(s)} \left(\delta_{qM}^{\tilde{X}(t)}(j-i+1) + [\delta_{qM(1+s)}^{\tilde{X}(t)}(j-i+1)]^s \right).$$

Note that if two sequences a_i, b_i with $a_i = b_i = 0$ for $i < 0$ obey $a_i, b_i = O(i^{-(1+\gamma)})$ then the convolution $c_j = \sum_{i=1}^{\infty} a_i b_{j-i+1}$ still obeys $c_j = O(j^{-(1+\gamma)})$ due to

$$|c_j| \leq \sum_{i=1, i \geq (j+1)/2}^{j+1} |a_i| \cdot |b_{j-i+1}| + \sum_{i=1, |j-i| \geq (j+1)/2}^{j+1} |a_i||b_{j-i+1}|$$

$$\leq \left(\frac{j+1}{2}\right)^{-(1+\gamma)} \sum_{i=1}^{j+1} |b_{j-i+1}| + \left(\frac{j+1}{2}\right)^{-(1+\gamma)} \sum_{i=1}^{j+1} |a_i| = O(j^{-(1+\gamma)}).$$

Together with Assumption (A7) and (8.50), (8.51) <u>or</u> (in case (*)) Assumption 7.3(A7') and (8.52), this shows $\sup_{t \in [0,1]} \delta_r^{g(\tilde{Z}(t),\theta)}(j) = O(j^{-(1+\gamma)})$.

The proof for (ii),(iii) is the same since

$$\left| \sup_{t,\eta} |M_i(t,\eta,u)| - \sup_{t,\eta} |M_i(t,\eta,u)^*| \right| \leq \sup_{t,\eta} |M_i(t,\eta,u) - M_i(t,\eta,u)^*|$$

$$\leq |\hat{K}|_\infty \sup_\theta |g(\tilde{Z}_i(u),\theta) - g(\tilde{Z}_i(u)^*,\theta)|$$

and (since $|d_u(t)|_\infty \leq \sup_s |\theta''(s)|_\infty \cdot b_n^2$ if $|t - u| \leq b_n$), for each $l = 1, \ldots, k$,

$$\left| \sup_t |\tilde{M}_i^{(2)}(t,u)_l| - \sup_t |\tilde{M}_i^{(2)}(t,u)_l^*| \right| \leq \sup_t |M_i^{(2)}(t,u)_l - M_i^{(2)}(t,u)_l^*|$$

$$\leq |\hat{K}|_\infty \sup_s |\theta''(s)|_\infty b_n^2$$

$$\times \sup_t \int_0^1 |g(\tilde{Z}_i(u),\theta(t) + sd_u(t)) - g(\tilde{Z}_i(u)^*,\theta(t) + sd_u(t))| ds$$

$$\leq |\hat{K}|_\infty \sup_s |\theta''(s)|_\infty b_n^2 \sup_{\theta \in \Theta} |g(\tilde{Z}_i(u),\theta) - g(\tilde{Z}_i(u)^*,\theta)|.$$

$\square$

LEMMA 8.8 (for tvGARCH).   *Let $q \geq 1$. Suppose that Assumption 7.3(A5'), (A7') hold with some $r > q$. For $s > 0$, let $\chi^{(s)} = (\chi_i^{(s)})_{i \in \mathbb{N}}$ be such that $\chi_i^{(s)} = O(i^{-(1+\gamma)})$. Let g be such that $\tilde{g}_{\tilde{\theta}}(y, x, \theta) := g(F(x, \tilde{\theta}, y), x, \theta)$ fulfills $\tilde{g} \in \mathcal{H}_{s,\iota}^{mult}(M_y, M_x, \chi^{(s)}, \bar{C}^{(s)})$ for all $s > 0$ small enough. Then*

(i) $\sup_{t \in [0,1]} \delta_q^{\sup_{|\theta - \theta(t)|_1 < \iota} |g(\tilde{Z}(t), \theta)|}(j) = O(j^{-(1+\gamma)})$.

(ii) *For n large enough,*

$$\sup_{u \in [0,1]} \sup_{t, |\eta - \eta_{b_n}(t)|_1 < \iota/2} \delta_q^{M(t,\eta,u)}(j) = O(j^{-(1+\gamma)}), \qquad \sup_{u \in [0,1]} \delta_q^{\sup_{t, |\eta - \eta_{b_n}(t)|_1 < \iota/2} |M(t,\eta,u)|}(j) = O(j^{-(1+\gamma)}).$$

(iii) *For n large enough, $\sup_{u \in [0,1]} \delta_q^{M^{(2)}(t,u)}(j) = O(b_n^2 j^{-(1+\gamma)})$, and $\sup_{u \in [0,1]} \delta_q^{\sup_t |M^{(2)}(t,u)|}(j) = O(b_n^2 j^{-(1+\gamma)})$.*

PROOF OF LEMMA 8.8.   (i) Let $\tilde{Z}_j(t)^*$ be a coupled version of $\tilde{Z}_j(t)$ where ζ_0 is replaced by ζ_0^*. By Lemma 8.5 we obtain that with some constant $\tilde{C} > 0$:

$$\delta_q^{\sup_{|\theta - \theta(t)|_1 < \iota} |g(\tilde{Z}(t), \theta)|}(j)$$
$$\leq \quad \|\sup_{|\theta - \theta(t)|_1 < \iota} |\tilde{g}_{\theta(t)}(\zeta_j, \tilde{X}_j(t), \theta) - \tilde{g}_{\theta(t)}(\zeta_j, \tilde{X}_j(t)^*, \theta)|\|_q$$
$$\leq \quad \tilde{C} \sum_{i=1}^{\infty} \chi_i \|\tilde{X}_{j-i+1}(t) - \tilde{X}_{j-i+1}(t)^*\|_{qM}$$
$$\leq \quad \tilde{C} \sum_{i=1}^{\infty} \chi_i \delta_{qM}^{\tilde{X}(t)}(j - i + 1).$$

The result now follows as in the proof of Lemma 8.7(i) with Assumption 7.3(A7').

(ii) We have for n large enough that

$$|\eta - \eta_{b_n}(t)|_1 = |\eta_1 - \theta(t)|_1 + |\eta_2 - b_n \theta'(t)|_1 < \iota/2 \quad \text{implies} \quad |(\eta_1 + \eta_2(u-t)b_n^{-1}) - \theta(t)|_1 \leq |\eta_1 - \theta(t)|_1 + |\eta_2|_1 < \iota$$

and $|\theta - \theta(t)|_1 < \iota$, $|u - t| \leq b_n$ implies $|\theta - \theta(u)|_1 < \iota$ due to uniform continuity of $\theta(\cdot)$. Therefore, we have for n large enough:

$$\left| \sup_{t, |\eta - \eta_{b_n}(t)|_1 < \iota/2} |M_i(t, \eta, u)| - \sup_{t, |\eta - \eta_{b_n}(t)|_1 < \iota/2} |M_i(t, \eta, u)^*| \right|$$
$$\leq \quad \sup_{t, |\eta - \eta_{b_n}(t)|_1 < \iota/2} |\hat{K}_{b_n}(u - t)| \cdot |g(\tilde{Z}_i(u), \eta_1 + \eta_2(u-t)b_n^{-1}) - g(\tilde{Z}_i(u), \eta_1 + \eta_2(u-t)b_n^{-1})|$$
$$\leq \quad \sup_{t, |\theta - \theta(t)|_1 < \iota} |\hat{K}_{b_n}(u - t)| \cdot |\tilde{g}_{\theta(u)}(\zeta_i, \tilde{X}_i(u), \theta) - \tilde{g}_{\theta(u)}(\zeta_i, \tilde{X}_i(u)^*, \theta)|$$
$$\leq \quad |\hat{K}|_\infty \cdot \sup_{|\theta - \theta(u)|_1 < \iota} |\tilde{g}_{\theta(u)}(\zeta_i, \tilde{X}_i(u), \theta) - \tilde{g}_{\theta(u)}(\zeta_i, \tilde{X}_i(u)^*, \theta)|.$$

The rest works as in (i).

(iii) For n large enough, it holds that $|u-t| \leq b_n$ implies that $\sup_{s\in[0,1]} |\theta(t)+sd_u(t)-\theta(u)| < \iota$ due to uniform continuity of $\theta(\cdot)$. Thus

$$\Big| \sup_t |M_i^{(2)}(t,u)| - \sup_t |M_i^{(2)}(t,u)^*| \Big|$$

$$\leq |\hat{K}|_\infty \sup_s |\theta''(s)|_\infty b_n^2 \sup_{|\theta-\theta(u)|<\iota} |\tilde{g}_{\theta(u)}(\zeta_i, \tilde{X}_i(u), \theta) - \tilde{g}_{\theta(u)}(\zeta_i, \tilde{X}_i(u)^*, \theta)|.$$

The rest works as in (i). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

LEMMA 8.9 (Lipschitz properties of $\hat{G}_n$). *Let $s \geq 0$.*

(i) *Let $g \in \mathcal{H}_s(M_y, M_x, \chi, \bar{C})$. Let Assumption 7.1(A5) hold with $r \geq 1 + s$. Then there exists some constant $\tilde{C} > 0$ such that*

$$\sup_{t\in[0,1]} \Big\| \sup_{\eta\neq\eta'} \frac{|\hat{G}_n(t,\eta) - \hat{G}_n(t,\eta')|}{|\eta-\eta'|_1} \Big\|_1 \leq \tilde{C},$$

and

$$\Big\| \sup_{t\neq t'} \sup_{\eta\neq\eta'} \frac{|\hat{G}_n(t,\eta) - \hat{G}_n(t',\eta')|}{|t-t'|+|\eta-\eta'|_1} \Big\|_1 \leq \tilde{C}b_n^{-2},$$

(ii) *(for tvGARCH) Let g be such that $\tilde{g}_{\tilde{\theta}}(y,x,\theta) := g(F(x,\tilde{\theta},y),x,\theta)$ fulfills $\tilde{g} \in \mathcal{H}_{s,\iota}^{mult}(M_y, M_x, \chi^{(s)}, \bar{C}^{(s)})$ with $\chi_i^{(s)} = O(i^{-(1+\gamma)})$. Let Assumption 7.3(A5') hold with $r \geq 1+s$ and let $\theta(\cdot)$ be continuous. Then there exists some constant $\tilde{C}^{(s)} > 0$ such that*

$$\sup_{t\in[0,1]} \Big\| \sup_{\substack{\eta\neq\eta' \\ |\eta-\eta_{b_n}(t)|_1<\iota/2, |\eta'-\eta_{b_n}(t)|_1<\iota/2}} \frac{|\hat{G}_n(t,\eta) - \hat{G}_n(t,\eta')|}{|\eta-\eta'|_1} \Big\|_1 \leq \tilde{C}^{(s)},$$

and

$$\Big\| \sup_{\substack{t\neq t'}} \sup_{\substack{\eta\neq\eta' \\ |\eta-\eta_{b_n}(t)|_1<\iota/2, |\eta'-\eta_{b_n}(t')|_1<\iota/2}} \frac{|\hat{G}_n(t,\eta) - \hat{G}_n(t',\eta')|}{|t-t'|+|\eta-\eta'|_1} \Big\|_1 \leq \tilde{C}^{(s)}b_n^{-2},$$

PROOF OF LEMMA 8.9. By Lemma 8.3(i), $\sup_{t\in[0,1]} \|R_{M_y,M_x}(\tilde{Z}_0(t))^{1+s}\|_1 < \infty$. This is needed several times in the following.

(i) Since $g \in \mathcal{H}_s(M_y, M_x, \chi, \bar{C})$ and $|i/n - t| \leq b_n$ inside the sum, it holds that
(8.53)

$$|\hat{G}_n(t,\eta) - \hat{G}_n(t,\eta')| \leq \bar{C}|\eta-\eta'|_1 \cdot (nb_n)^{-1} \sum_{i=1}^n |\hat{K}_{b_n}(i/n-t)|\{R_{M_y,M_x}(\tilde{Z}_i(i/n))^{1+s} + \|R_{M_y,M_x}(\tilde{Z}_i(i/n))^{1+s}\|_1\}.$$

Furthermore, $(nb_n)^{-1} \sum_{i=1}^n |\hat{K}_{b_n}(i/n - t)| \le |\hat{K}|_\infty$. This yields the assertion.

Since $g \in \mathcal{H}_s(M_y, M_x, \chi, \tilde{C})$, we have with some constant $\tilde{C} > 0$:

$$|\hat{G}_n(t, \eta) - \hat{G}_n(t', \eta')|$$

$$\le \quad (nb_n)^{-1} \sum_{i=1}^n |\hat{K}_{b_n}(i/n - t) - \hat{K}_{b_n}(i/n - t')| \cdot \sup_\theta \{|g(\tilde{Z}_i(i/n), \theta)| + \|g(\tilde{Z}_i(i/n), \theta)\|_1\}$$

$$+(nb_n)^{-1} \sum_{i=1}^n |\hat{K}_{b_n}(i/n - t')| \cdot |g(\tilde{Z}_i(i/n), \eta_1 + \eta_2(i/n - t)b_n^{-1}) - g(\tilde{Z}_i(i/n), \eta_1' + \eta_2'(i/n - t')b_n^{-1})$$

$$\le \quad [b_n^{-2} L_{\hat{K}}|t - t'| + b_n^{-1}|\hat{K}|_\infty \{|\eta - \eta'|_1 + |\eta_2| \cdot |t - t'|b_n^{-1}] \cdot \frac{1}{n} \sum_{i=1}^n \{R_{M_y, M_x}(\tilde{Z}_i(i/n))^{1+s}$$

$$+ \|R_{M_y, M_x}(\tilde{Z}_i(i/n))^{1+s}\|_1\}.$$

Since $\sup_{\eta \in E_n} |\eta_2|_1 < \infty$ is compact, this gives the result.

(ii) We now have

$$|\hat{G}_n(t, \eta) - \hat{G}_n(t, \eta')|$$

$$\le \quad (nb_n)^{-1} \sum_{i=1}^n |\hat{K}_{b_n}(i/n - t)| \cdot \left| \tilde{g}_{\theta(i/n)}(\zeta_i, \tilde{X}_i(i/n), \eta_1 + \eta_2(i/n - t)b_n^{-1}) \right.$$

$$\left. -\tilde{g}_{\theta(i/n)}(\zeta_i, \tilde{X}_i(i/n), \eta_1' + \eta_2'(i/n - t)b_n^{-1}) \right|.$$

Here, $|\eta - \eta_{b_n}(t)| < \iota/2$ implies $|(\eta_1 + \eta_2(i/n - t)b_n^{-1}) - \theta(t)| < \iota$ for n large enough. Since $\theta(\cdot)$ is uniformly continuous, $|\theta - \theta(t)|_1 < \iota$, $|i/n - t| \le b_n$ imply $|\theta - \theta(i/n)|_1 < \iota$ for n large enough. Since $\tilde{g}_{\tilde{\theta}} \in \mathcal{H}_{s,\iota}^{mult}(M_y, M_x, \chi^{(s)}, \bar{C}^{(s)})$, we obtain

$$|\hat{G}_n(t, \eta) - \hat{G}_n(t, \eta')| \quad \le \quad \bar{C}^{(s)}|\eta - \eta'|_1(nb_n)^{-1} \sum_{i=1}^n |\hat{K}_{b_n}(i/n - t)| \cdot \{R_{M,M}(1, \tilde{X}_i(i/n))^{1+s}(1 + |\zeta_i|^M)^{1+s}$$

$$+ \|R_{M,M}(1, \tilde{X}_i(i/n))^{1+s}(1 + |\zeta_i|^M)^{1+s}\|_1\},$$

giving the result.

We have

$$|\hat{G}_n(t, \eta) - \hat{G}_n(t', \eta')|$$

$$\le \quad (nb_n)^{-1} \sum_{i=1}^n |\hat{K}_{b_n}(i/n - t) - \hat{K}_{b_n}(i/n - t')|$$

$$\times \sup_{|\eta - \eta_{b_n}(t)| < \iota/2} \{|\tilde{g}_{\theta(i/n)}(\zeta_i, \tilde{X}_i(i/n), \eta_1 + \eta_2(i/n - t)b_n^{-1})| + \|\tilde{g}_{\theta(i/n)}(\zeta_i, \tilde{X}_i(i/n), \eta_1 + \eta_2(i/n - t)b_n^{-1})\|_1\}$$

$$+(nb_n)^{-1} \sum_{i=1}^n |\hat{K}_{b_n}(i/n - t')| \cdot |\tilde{g}_{\theta(i/n)}(\zeta_i, \tilde{X}_i(i/n), \eta_1 + \eta_2(i/n - t)b_n^{-1})$$

$$-\tilde{g}_{\theta(i/n)}(\zeta_i, \tilde{X}_i(i/n), \eta_1' + \eta_2'(i/n - t')b_n^{-1}).$$

The same argumentation as before allows us to use the Lipschitz properties of $\tilde{g}_{\theta(i/n)}$ w.r.t. θ, giving the result. $\qquad\square$

For the proof of the following lemma, we will make use of the adjusted dependence measure $\|\cdot\|_{q,\alpha}$ which is defined as follows (cf. [57]): For some zero-mean random variable Z, let $\|Z\|_{q,\alpha} := \sup_{m\geq 0}(m+1)^{\alpha}\Delta_q^Z(m)$.

LEMMA 8.10. *Let $\gamma > 1$. For $s \geq 0$, let $\chi_i^{(s)} = (\chi_i^{(s)})_{i\in\mathbb{N}}$ be a sequence with $\chi_i^{(s)} = O(i^{-(1+\gamma)})$. Recall the notation from (8.1). Assume that either (in the case (a)) Assumption 7.1(A5), (A6), (A7) or (in the cases (b),(c)) Assumption 7.3(A5'), (A6'), (A7') hold with some r specified below.*

(i) *Let $r \geq 1+\varsigma$, $\varsigma \geq 0$ and assume either that $\varsigma = 0$ and $g \in \mathcal{H}_0(M_y, M_x, \chi^{(0)}, \bar{C}^{(0)})$ or $\varsigma > 0$ and for all $s > 0$ small enough, $g \in \mathcal{H}_s(M_y, M_x, \chi^{(s)}, \bar{C}^{(s)})$. Then*

$$\|\sup_{t\in(0,1)}\sup_{\eta\in E_n}|\hat{G}_n(t,\eta) - G_n^c(t,\eta)|\|_1 = O((nb_n)^{-1}).$$

(ii) *Fix $t \in [0,1]$ and assume that $nb_n \to \infty$. Let $r \geq 1+\varsigma$, $\varsigma > 0$.*
   (a) *If for all $s > 0$ small enough, $g \in \mathcal{H}_s(M_y, M_x, \chi^{(s)}, \bar{C}^{(s)})$, then*

$$\sup_{\eta\in E_n}|\hat{G}_n(t,\eta)| = o_{\mathbb{P}}(1).$$

   (b) *If for all $s > 0$ small enough, $\tilde{g}_{\tilde{\theta}}(y,x,\theta) := g(F(y,x,\tilde{\theta}),x,\theta)$ fulfills $\tilde{g} \in \mathcal{H}_{s,\iota}^{mult}(M,\chi^{(s)},\bar{C}^{(s)})$, then*

$$\sup_{|\eta-\eta_{b_n}(t)|<\iota}|\hat{G}_n(t,\eta)| = o_{\mathbb{P}}(1) \qquad if \quad b_n \to 0.$$

   (c) *If for all $s > 0$ small enough, g fulfills (7.6) and $g \in \mathcal{H}_s(2M_y, 2M_x, \chi^{(s)}, \bar{C}^{(s)})$, then*

$$\sup_{\eta\in E_n}|\hat{G}_n(t,\eta)| = o_{\mathbb{P}}(1).$$

(iii) *Let $r \geq 2+\varsigma$, $\varsigma > 0$. Define $\beta_n = \log(n)^{1/2}(nb_n)^{-1/2}b_n^{-1/2}$.*
   (a) *If for all $s > 0$ small enough, $g \in \mathcal{H}_s(M_y, M_x, \chi^{(s)}, \bar{C}^{(s)})$, then*

$$\sup_{t\in(0,1)}\sup_{\eta\in E_n}|\hat{G}_n(t,\eta)| = O_{\mathbb{P}}(\beta_n).$$

   (b) *If g is such that $\tilde{g}_{\tilde{\theta}}(y,x,\theta) := g(F(y,x,\tilde{\theta}),x,\theta)$ and for all $s > 0$ small enough, $\tilde{g} \in \mathcal{H}_{s,\iota}^{mult}(M,\chi^{(s)},\bar{C}^{(s)})$, then*

$$\sup_{t\in(0,1)}\sup_{|\eta-\eta_{b_n}(t)|_1<\iota}|\hat{G}(t,\eta)| = O_{\mathbb{P}}(\beta_n).$$

(c) If for all $s > 0$ small enough, g fulfills (7.6) and $g \in \mathcal{H}_s(2M_y, 2M_x, \chi^{(s)}, \bar{C}^{(s)})$, then

$$\sup_{t \in (0,1)} \sup_{\eta \in E_n} |\hat{G}(t, \eta)| = O_{\mathbb{P}}(\beta_n).$$

PROOF OF LEMMA 8.10. We abbreviate $\chi = \chi^{(s)}$ and $\bar{C} = \bar{C}^{(s)}$.
(i) By Lemma 8.3(i),(ii) and by Assumption 7.1(A6), we obtain (independent of (7.4) or (7.5)) that for some $C > 0$:

$$\|\sup_{\theta \in \Theta} |g(Z_i, \theta) - g(Z_i^c, \theta)|\|_1 \leq C \sum_{j=0}^{\infty} \hat{\chi}_j \|Z_{ij} - Z_{ij}^c\|_M \leq 2C \sum_{j=i}^{\infty} \chi_j \|Z_{ij}\|_M \leq 2CD \sum_{j=i}^{\infty} \chi_j.$$

Similarly, we have in the case (7.4) that

$$\|\sup_{\theta \in \Theta} |g(Z_i, \theta) - g(\tilde{Z}_i(i/n), \theta)|\|_1 \leq C\left(\|Y_i - \tilde{Y}_i(i/n)\|_M + \sum_{j=1}^{\infty} \chi_j \|X_{ij} - \tilde{X}_{ij}(i/n)\|_M\right)$$

$$\leq CC_A |\chi|_1 n^{-1},$$

while in the case (7.5) there exists $C_2 > 0$ such that

$$\|\sup_{\theta \in \Theta} |g(Z_i, \theta) - g(\tilde{Z}_i(i/n), \theta)|\|_1 \leq C_2 \sum_{j=1}^{\infty} \chi_j \|X_{ij} - \tilde{X}_{ij}(i/n)\|_M \leq C_2 C_A |\chi|_1 n^{-1}.$$

Thus

$$\|\sup_{t \in (0,1)} \sup_{\eta \in E_n} |G_n(t, \eta) - G_n^c(t, \eta)|\|_1$$

$$\leq |K|_\infty (nb_n)^{-1} \sum_{i=1}^{n} \|\sup_{\theta \in \Theta} |g(Z_i, \theta) - g(Z_i^c, \theta)|\|_1$$

$$\leq 2CD|K|_\infty (nb_n)^{-1} \sum_{i=1}^{n} \sum_{j=i}^{\infty} \chi_j + |K|_\infty (C \vee C_2) |\chi|_1 (nb_n)^{-1}.$$

Since $\chi_j = O(j^{-1+\gamma})$, it holds that $\sum_{i=1}^{n} \sum_{j=i}^{\infty} \chi_j = O(1)$ and the assertion is proved. The proofs under Assumption 7.3 are similar in view of Lemma 8.4.

(ii) (a) Fix $Q > 0$. Let $\kappa > 0$. Let $E_n^{(\kappa)}$ be a discretization of E_n such that for each $\eta \in E_n$ one can find $\eta' \in E_n^{(\kappa)}$ with $|\eta - \eta'|_1 \leq \kappa$. Note that $\#E_n^{(\kappa)}$ does not need to depend on n. Then

$$\mathbb{P}\left(\sup_{\eta \in E_n} |\hat{G}_n(t, \eta)| > Q\right) \leq \#E_n^{(\kappa)} \sup_{\eta \in E_n} \mathbb{P}\left(|\hat{G}_n(t, \eta)| > Q/2\right)$$

$$(8.54) \qquad\qquad\qquad + \mathbb{P}\left(\sup_{|\eta - \eta'|_1 \leq \kappa} |\hat{G}_n(t, \eta) - \hat{G}_n(t, \eta')| > Q/2\right).$$

By Markov's inequality, we have for $0 \leq s \leq \varsigma$,

$$\mathbb{P}\big(|\hat{G}_n(t,\eta)| > Q/2\big) \leq \frac{\|\hat{G}_n(t,\eta)\|_{1+s}^{1+s}}{(Q/2)^{1+s}}.$$

Using Burkholder's moment inequality (cf. [7]) and Lemma 8.7(i) applied for $q = 1 + s$, $s > 0$ small enough, the computation

$$(8.55)\|\hat{G}_n(t,\eta)\|_{1+s}$$

$$\leq (nb_n)^{-1} \sum_{l=0}^{\infty} \Big\| \sum_{i=1}^{n} \hat{K}_{b_n}(i/n - t) P_{i-l} g(\tilde{Z}_i(i/n), \eta_1 + \eta_2(i/n - t)b_n^{-1}) \Big\|_{1+s}$$

$$\leq s^{-1}(nb_n)^{-1} \sum_{l=0}^{\infty} \Big(\Big\| \sum_{i=1}^{n} \hat{K}_{b_n}(i/n - t)^2 P_{i-l}^2 g(\tilde{Z}_i(i/n), \eta_1 + \eta_2(i/n - t)b_n^{-1}) \Big\|_{(1+s)/2}^{(1+s)/2} \Big)^{1/(1+s)}$$

$$\leq s^{-1}(nb_n)^{-s/(1+s)}|\hat{K}|_{\infty} \sum_{l=0}^{\infty} \sup_{t\in[0,1]} \delta_{1+s}^{\sup_{\theta\in\Theta}|g(\tilde{Z}(t),\theta)|}(l) = O((nb_n)^{-s/(1+s)}),$$

shows that the first summand in (8.54) tends to zero. For the second summand, Lemma 8.9(i) implies

$$\mathbb{P}\big(\sup_{|\eta-\eta'|_1 \leq \kappa} |\hat{G}_n(t,\eta) - \hat{G}_n(t,\eta')| > Q/2 \big) \leq \frac{2\tilde{C}\kappa}{Q},$$

which can be made arbitrary small by choosing ι small enough. So we have shown that (8.54) tends to zero for $n \to \infty$.

(b) The proof is similar to (a) by using 8.9(ii) and Lemma 8.8(i) instead of Lemma 8.9(i) and Lemma 8.7(i).

(c) The proof is similar to (a) by using Lemma 8.7(i)(*) instead of Lemma 8.7(i).

(iii) (a) We use a chaining argument. Let $r = n^3$ and let $E_{n,r}$ be a discretization of E_n such that for each $\eta \in E_n$ one can find $\eta' \in E_{n,r}$ with $|\eta - \eta'| \leq r^{-1}$. Define $\mathcal{T}_{n,r} := \{i/r : i = 1, \ldots, r\}$ as a discretization of $(0,1)$. Then $\#(E_{n,r} \times \mathcal{T}_{n,r}) = O(r^{2d\Theta+1})$. For some constant $Q > 0$, we have

$$\mathbb{P}\Big(\sup_{\eta \in E_n, t \in (0,1)} |\hat{G}_n(t,\eta)| > Q\beta_n \Big)$$

$$\leq \mathbb{P}\Big(\sup_{\eta \in E_{n,r}, t \in \mathcal{T}_{n,r}} |\hat{G}_n(t,\eta)| > Q\beta_n/2 \Big)$$

$$(8.56) \qquad + \mathbb{P}\Big(\sup_{|\eta-\eta'| \leq r^{-1}, |t-t'| \leq r^{-1}} |\hat{G}_n(t,\eta) - \hat{G}_n(t',\eta')| > Q\beta_n/2 \Big).$$

Let $\alpha = 1/2$. Let $M_i(t,\eta,u) := \hat{K}_{b_n}(u - t)g(\tilde{Z}_i(u), \eta_1 + \eta_2(u - t)b_n^{-1})$. By Lemma 8.7(ii) applied with $q = 2 + s$, $s > 0$ small enough, we have $\sup_u \Delta_{2+s}^{\sup_{t,\eta}|M(t,\eta,u)|}(k) = O(k^{-(1+\gamma)})$.

Thus

$$W_{2+s,\alpha} \quad := \quad \sup_{u\in[0,1]} \|\sup_{t,\eta} |M_i(t,\eta,u)|\|_{2+s,\alpha} = \sup_{m\geq0}(m+1)^\alpha \sup_{u\in[0,1]} \sup_{t,\eta} \Delta_{2+s}^{\sup_{t,\eta}|M(t,\eta,u)|}(m) < \infty.$$

(independent of n) and

$$W_{2,\alpha} \quad := \quad \sup_{u\in[0,1]} \sup_{t,\eta} \|M_i(t,\eta,u)\|_{2,\alpha} = \sup_{m\geq0}(m+1)^\alpha \sup_{u\in[0,1]} \sup_{t,\eta} \Delta_2^{M(t,\eta,u)}(m) < \infty$$

(independent of n). Note that $l = 1 \wedge \log\#(E_{n,r} \times \mathcal{T}_{n,r}) \leq 3(2d_\Theta+1)\log(n)$ and $Q\beta_n(nb_n) = Qn^{1/2}\log(n)^{1/2} \geq \sqrt{nl}W_{2,\alpha} + n^{1/(2+s)}l^{3/2}W_{2+s,\alpha} \gtrsim n^{1/2}\log(n)^{1/2} + n^{1/(2+s)}\log(n)^{3/2}$ for Q large enough. By applying Theorem 6.2 of [57] (the proof therein also works for the uniform functional dependence measure) with $q = 2+s$ and $\alpha = 1/2$ to $(M_i(t,\eta,i/n))_{t\in\mathcal{T}_{n,r},\eta\in E_{n,r}}$, we have with some constant $C_\alpha > 0$:

$$\mathbb{P}\Big(\sup_{\eta'\in E_{n,r},t'\in\mathcal{T}_{n,r}} |\hat{G}_n(t',\eta')| \geq Q\beta_n/2\Big)$$

$$\leq \quad \frac{C_\alpha n \cdot l^{1+s/2}W_{2+s,\alpha}^{2+s}}{(Q/2)^{2+s}(\delta_n(nb_n))^{2+s}} + C_\alpha\exp\Big(-\frac{C_\alpha(Q/2)^2(\beta_n(nb_n))^2}{nW_{2,\alpha}^2}\Big)$$

$$\lesssim \quad n^{-s/2} + \exp\Big(-\frac{(nb_n)b_n^{-1}\log(n)}{n}\Big)$$

$$(8.57) \qquad\qquad \to \quad 0.$$

By Markov's inequality and Lemma 8.9(i),

$$(8.58) \qquad \mathbb{P}\Big(\sup_{|\eta-\eta'|_1\leq r^{-1},|t-t'|\leq r^{-1}} |\hat{G}_n(t,\eta) - \hat{G}_n(t',\eta')| \geq C\beta_n/2\Big) = O\Big(\frac{b_n^{-2}r^{-1}}{\beta_n}\Big).$$

We have $b_n^{-2}r^{-1}\beta_n^{-1} = b_n^{-2}n^{-3}(nb_n)^{1/2}b_n^{1/2}\log(n)^{-1/2} \to 0$. Inserting (8.57) and (8.58) into (8.56), we obtain the result.
(b) The proof is similar to (a) by using 8.9(ii) and Lemma 8.8(ii) instead of Lemma 8.9(i) and Lemma 8.7(ii).
(c) The proof is similar to (a) by using 8.9(ii)(*) instead of Lemma 8.9(ii).   □

LEMMA 8.11.   *For $g : \mathbb{R} \times \mathbb{R}^\mathbb{N} \times \Theta \to \mathbb{R}$. Let*

$$\hat{B}_n(t,\eta) = (nb_n)^{-1}\sum_{i=1}^n \hat{K}_{b_n}(i/n - t)g(\tilde{Z}_i(i/n),\eta_1 + \eta_2(i/n - t)b_n^{-1}).$$

(a) *If Assumption 7.1(A5) is fulfilled with $r \geq 1+s$, $s \geq 0$ and $g \in \mathcal{H}_s(M_y, M_x, \chi, \bar{C})$, then*

$$\sup_{t\in(0,1)} \sup_{\eta\in E_n} |\mathbb{E}\hat{B}_n(t,\eta) - \int_{-t/b_n}^{(1-t)/b_n} \hat{K}(x)\mathbb{E}g(\tilde{Z}_0(t),\eta_1 + \eta_2x)dx| = O((nb_n)^{-1} + b_n).$$

(b) If Assumption 7.3(A5') is fulfilled with $r \geq 1 + s$ and g is such that $\tilde{g}_{\tilde{\theta}}(y, x, \theta) := g(F(y, x, \tilde{\theta}), x, \theta)$ fulfills $\tilde{g} \in \mathcal{H}_{s,\iota}^{mult}(M, \chi, \bar{C})$, then

$$\sup_{t \in (0,1)} \sup_{|\eta - \eta_{b_n}(t)| < \iota} |\mathbb{E}\hat{B}_n(t, \eta) - \int_{-t/b_n}^{(1-t)/b_n} \hat{K}(x)\mathbb{E}g(\tilde{Z}_0(t), \eta_1 + \eta_2 x)dx| = O((nb_n)^{-1} + b_n).$$

If the supremum is taken over $t \in \mathcal{T}_n$ instead of $t \in (0, 1)$, then $\int_{-t/b_n}^{(1-t)/b_n}$ can be replaced by $\int_{-1}^{1}$.

PROOF OF LEMMA 8.11. (a) Let $\tilde{B}_n(t, \eta) := (nb_n)^{-1} \sum_{i=1}^{n} \hat{K}_{b_n}(i/n - t)g(\tilde{Z}_i(t), \eta_1 + \eta_2(i/n - t)b_n^{-1})$. By Lemma 8.3(i), we have with some constant $\tilde{C} > 0$ that either in the case of (7.4),

$$\|g(\tilde{Z}_0(i/n), \eta_1 + \eta_2(i/n - t)b_n^{-1}) - g(\tilde{Z}_0(t), \eta_1 + \eta_2(i/n - t)b_n^{-1})\|_1$$
$$\leq \tilde{C}\left(\|\tilde{Y}_0(i/n) - \tilde{Y}_0(t)\|_M + \sum_{i=1}^{\infty} \chi_i \|\tilde{X}_{-i}(i/n) - \tilde{X}_{-i}(t)\|_M\right) \leq \tilde{C}C_B(1 + |\chi|_1)b_n$$

or in the case of (7.5),

$$\|g(\tilde{Z}_0(i/n), \eta_1 + \eta_2(i/n - t)b_n^{-1}) - g(\tilde{Z}_0(t), \eta_1 + \eta_2(i/n - t)b_n^{-1})\|_1$$
$$\leq \tilde{C}\sum_{i=1}^{\infty} \chi_i \|\tilde{X}_{-i}(i/n) - \tilde{X}_{-i}(t)\|_M \leq \tilde{C}C_B|\chi|_1 b_n.$$

Thus

$$\|\hat{B}_n(t, \eta) - \tilde{B}_n(t, \eta)\|_1$$
$$\leq (nb_n)^{-1} \sum_{i=1}^{n} |\hat{K}_{b_n}(i/n - t)|$$
$$\times \|g(\tilde{Z}_i(i/n), \eta_1 + \eta_2(i/n - t)b_n^{-1}) - g(\tilde{Z}_i(t), \eta_1 + \eta_2(i/n - t)b_n^{-1})\|_1$$
$$\leq \tilde{C}|\hat{K}|_{\infty}C_B(1 + |\chi|_1)b_n.$$

Since $\hat{K}$ is of bounded variation and $\theta \mapsto \mathbb{E}g(\tilde{Z}_0(t), \theta)$ is Lipschitz continuous due to $g \in \mathcal{H}(M_y, M_x, \chi, \bar{C})$ and Lemma 8.3, a Riemannian sum argument yields

$$\tilde{B}_n(t, \eta) = (nb_n)^{-1} \sum_{i=1}^{n} \hat{K}_{b_n}(i/n - t)\mathbb{E}g(\tilde{Z}_0(t), \eta_1 + \eta_2(i/n - t)b_n^{-1})$$
$$= \int_{-t/b_n}^{(1-t)/b_n} \hat{K}(x)\mathbb{E}g(\tilde{Z}_0(t), \eta_1 + \eta_2 x)dx + O((nb_n)^{-1}),$$

uniformly in $t \in (0, 1)$, $\eta \in E_n$.
(b) The proof is the same by using Lemma 8.5 with $q = 1$ instead of Lemma 8.3. □

LEMMA 8.12.   *Let $\eta_{b_n}(t) = (\theta(t)^{\mathsf{T}}, b_n\theta'(t)^{\mathsf{T}})^{\mathsf{T}}$. Let Assumption 7.1 hold with $r = 1$ or let Assumption 7.3 hold with $r = 2 + \varsigma, \varsigma > 0$.*

(i) *Then uniformly in $t \in \mathcal{T}_n$,*

$$(8.59) \qquad \mathbb{E}\nabla_{\eta_1}\hat{L}_{n,b_n}^{\circ}(t, \eta_{b_n}(t)) = b_n^2 \frac{\mu_{K,2}}{2} V(t)\theta''(t) + O(b_n^3 + (nb_n)^{-1}).$$

Furthermore, it holds uniformly in $t \in (0, 1)$ that
(8.60)

$$\mathbb{E}\nabla_{\eta}\hat{L}_{n,b_n}^{\circ}(t, \eta_{b_n}(t)) = \frac{b_n^2}{2}\int_{-t/b_n}^{(1-t)/b_n} K(x)\begin{pmatrix} x^2 \\ x^3 \end{pmatrix} dx \otimes [V(t)\theta''(t)] + O(b_n^3 + (nb_n)^{-1}).$$

(ii) *If Assumptions 7.1, 7.2 or Assumptions 7.3, 7.4 hold with the r specified above, then uniformly in $t \in \mathcal{T}_n$,*

$$\mathbb{E}\nabla_{\eta_2}\hat{L}_{n,b_n}^{\circ}(t, \eta_{b_n}(t)) = b_n^3 \frac{\mu_{K,4}}{2} V(t)\,bias(t) + O(b_n^4 + (nb_n)^{-1}),$$

where $bias(t) = \frac{1}{3}\theta^{(3)}(t) + V(t)^{-1}\mathbb{E}[\partial_t\nabla_\theta^2\ell(\tilde{Z}_0(t), \theta(t))]\cdot\theta''(t)$, and the term $O(b_n^3)$ in (8.59) can be replaced by $O(b_n^4)$.

PROOF OF LEMMA 8.12.   (i) Let $U_{i,n}(t) = (K_{b_n}(i/n - t), K_{b_n}(i/n - t)(i/n - t)b_n^{-1})^{\mathsf{T}}$. By a Taylor expansion of $\theta(i/n)$ around t, we have

$$\theta(i/n) = \theta(t) + \theta'(t)(i/n - t) + r_n(t),$$

where $r_n(t) = \theta''(t)\frac{(i/n-t)^2}{2} + \theta'''(\tilde{t})\frac{(i/n-t)^3}{6}$ and $\tilde{t}$ is between t and i/n. We conclude that

$$\nabla_{\eta}\hat{L}_{n,b_n}^{\circ}(t, \eta_{b_n}(t)) - (nb_n)^{-1}\sum_{i=1}^{n} U_{i,n}(t) \otimes \nabla_\theta\ell(\tilde{Z}_i(i/n), \theta(i/n))$$

$$(8.61) \qquad = (nb_n)^{-1}\sum_{i=1}^{n} U_{i,n}(t) \otimes \Big\{ \int_0^1 \nabla_\theta^2\ell(\tilde{Z}_i(i/n), \theta(i/n) + sr_n(t))ds \cdot r_n(t) \Big\}.$$

Using $\nabla_\theta^2\ell \in \mathcal{H}(M_y, M_x, \chi, \bar{C})$ (if Assumption 7.1 holds) or $\nabla_\theta^2\ell \in \mathcal{H}_s(2M_y, 2M_x, \chi, \bar{C})$ with $s > 0$ small enough (if Assumption 7.3 holds), we obtain with Lemma 8.3 for $|i/n - t| \le b_n$:

$$(8.62) \qquad \|\nabla_\theta^2\ell(\tilde{Z}_i(i/n), \theta(i/n) + sr_n(t)) - \nabla_\theta^2\ell(\tilde{Z}_i(t), \theta(t))\|_1 = O(b_n + n^{-1}).$$

Using (8.61), $\mathbb{E}\nabla_\theta\ell(\tilde{Z}_i(i/n), \theta(i/n)) = 0$ (by Assumption 7.1(A1),(A3) or Assumption 7.3(A1'), (A3')) and (8.62), we obtain

$$\mathbb{E}\nabla_{\eta}\hat{L}_{n,b_n}^{\circ}(t, \eta_{b_n}(t))$$

$$= (nb_n)^{-1}\sum_{i=1}^{n} U_{i,n}(t) \otimes \Big\{\mathbb{E}\nabla_\theta^2\ell(\tilde{Z}_i(t), \theta(t)) \cdot \theta''(t)\frac{(i/n - t)^2}{2}\Big\} + O(b_n^3 + n^{-1})$$

$$(8.63) \qquad = \begin{pmatrix} b_n^2\frac{\mu_{K,2}}{2}V(t)\theta''(t) \\ 0 \end{pmatrix} + O(b_n^3 + n^{-1} + (nb_n)^{-1}),$$

which shows (8.59).

(8.60) follows by a more careful examination of the above Riemannian sum: Under Assumption 7.2, we have $r_n(t) = \theta''(t)\frac{(i/n-t)^2}{2} + \theta^{(3)}(t)\frac{(i/n-t)^3}{6} + \theta^{(4)}(\tilde{t})\frac{(i/n-t)^4}{24}$, where $\tilde{t}$ is between t and i/n. We now use a more precise Taylor argument as in (8.61). We have

$$
\nabla_\eta \hat{L}^\circ_{n,b_n}(t, \eta_{b_n}(t)) - (nb_n)^{-1}\sum_{i=1}^n U_{i,n}(t) \otimes \nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n))
$$

$$
(8.64) \quad = \quad (nb_n)^{-1}\sum_{i=1}^n U_{i,n}(t) \otimes \nabla_\theta^2 \ell(\tilde{Z}_i(i/n), \theta(i/n))r_n(t)
$$

$$
+ (nb_n)^{-1}\sum_{i=1}^n U_{i,n}(t) \otimes \Big\{ \int_0^1 \nabla_\theta^2 \ell(\tilde{Z}_i(i/n), \theta(i/n) + sr_n(t))
$$

$$
- \nabla_\theta^2 \ell(\tilde{Z}_i(i/n), \theta(i/n))ds \cdot r_n(t) \Big\}.
$$

Since $\nabla_\theta^2 \ell \in \mathcal{H}(M_y, M_x, \chi, \bar{C})$ (if Assumption 7.1 holds) or $\nabla_\theta^2 \ell \in \mathcal{H}_s(2M_y, 2M_x, \chi, \bar{C})$ for $s > 0$ small enough (if Assumption 7.3 holds), we have by Lemma 8.3:

$$
(8.65) \quad \big\| \nabla_\theta^2 \ell(\tilde{Z}_i(i/n), \theta(i/n) + sr_n(t)) - \nabla_\theta^2 \ell(\tilde{Z}_i(i/n), \theta(i/n)) \big\|_1 = O(r_n(t)) = O(|i/n-t|^2).
$$

This shows that the expectation of the second summand in (8.64) is $O(b_n^4)$. We now discuss the first term in (8.64). Put $v_i(t) := \nabla_\theta^2 \ell(\tilde{Z}_i(t), \theta(t))$. By Assumption 7.2, $t \mapsto v_i(t)$ is continuously differentiable. By Taylor's expansion, $v_i(i/n) = v_i(t) + (i/n - t)\partial_t v_i(t) + (i/n - t)\int_0^1 \partial_t v_i(t + s(i/n - t)) - \partial_t v_i(t)ds$. We have

$$
(8.66) \quad \mathbb{E}\Big[(nb_n)^{-1}\sum_{i=1}^n U_{i,n}(t) \otimes v_i(t)r_n(t)\Big] = \begin{pmatrix} b_n^2 \frac{\mu_{K,2}}{2} V(t)\theta''(t) \\ b_n^3 \frac{\mu_{K,4}}{6} V(t)\theta^{(3)}(t) \end{pmatrix} + O(n^{-1} + b_n^4),
$$

since K has bounded variation and $\int K(x)x^3 dx = 0$ by symmetry. Similarly,

$$
\mathbb{E}\Big[(nb_n)^{-1}\sum_{i=1}^n U_{i,n}(t) \otimes \partial_t v_i(t)r_n(t)\Big]
$$

$$
(8.67) \quad = \quad \begin{pmatrix} 0 \\ b_n^3 \frac{\mu_{K,4}}{2} \mathbb{E}[\partial_t \nabla_\theta^2 \ell(\tilde{Z}_0(t), \theta(t))]\theta''(t) \end{pmatrix} + O(n^{-1} + b_n^4).
$$

Finally, Lemma 8.6 applied to $g = \nabla_\theta^2 \ell$ (use Assumption 7.2 or 7.4) yields:

$$
(8.68) \quad \|\partial_t v_i(t + s(i/n - t)) - \partial_t v_i(t)\|_1 = O(|i/n - t|).
$$

The results (8.66), (8.67) and (8.68) imply

$$
\mathbb{E}\Big[(nb_n)^{-1}\sum_{i=1}^n U_{i,n}(t) \otimes \nabla_\theta^2 \ell(\tilde{Z}_i(i/n), \theta(i/n))r_n(t)\Big]
$$

$$
= \begin{pmatrix} b_n^2 \frac{\mu_{K,2}}{2} V(t)\theta''(t) \\ b_n^3 \mu_{K,4} \cdot \big\{ \frac{1}{6} V(t)\theta^{(3)}(t) + \frac{1}{2}\mathbb{E}[\partial_t \nabla_\theta^2 \ell(\tilde{Z}_0(t), \theta(t))] \cdot \theta''(t) \big\} \end{pmatrix} + O(n^{-1} + b_n^4),
$$

which together with (8.64) gives the result. □

LEMMA 8.13 (Lipschitz properties of Π_n). *Let $s \geq 0$. Suppose that Assumption 7.1 holds with $r \geq 1$ or Assumption 7.3 holds with $r > 1$. Define*

$$\Pi_n(t) := (nb_n)^{-1} \sum_{i=1}^{n} (M_i^{(2)}(t, i/n) - \mathbb{E}M_i^{(2)}(t, i/n)),$$

where

$$M_i^{(2)}(t, u) = \hat{K}_{b_n}(u - t) \cdot \int_0^1 M_i(t, u)ds \cdot d_u(t),$$

$M_i(u, t) = \nabla_\theta^2 \ell(\tilde{Z}_i(u), \theta(t) + sd_u(t))$ *and* $d_u(t) = \theta(u) - \theta(t) - (u - t)\theta'(t)$. *Then there exist come constants* $\tilde{C}, \iota' > 0$ *such that*

$$\left\| \sup_{t \neq t', |t-t'| < \iota'} \frac{|\Pi_n(t) - \Pi_n(t')|}{|t - t'|_1} \right\|_1 \leq \tilde{C}.$$

PROOF OF LEMMA 8.13. We have

$$\begin{aligned}
&|M_i^{(2)}(t, u) - M_i^{(2)}(t', u)| \\
\leq\quad &|\hat{K}_{b_n}(u - t) - \hat{K}_{b_n}(u - t')| \cdot |M_i(t, u)| \cdot |d_u(t)| \\
&+ |\hat{K}_{b_n}(u - t')| \cdot |M_i(t, u) - M_i(t', u)| \cdot |d_u(t)| \\
&+ |\hat{K}_{b_n}(u - t')| \cdot |M_i(t', u)| \cdot |d_u(t) - d_u(t')|.
\end{aligned}$$

If Assumption 7.1 holds, we have

$$\begin{aligned}
|M_i(t, u)| &\leq \sup_{\theta \in \Theta} |g(\tilde{Z}_i(u), \theta)|, \\
|M_i(t, u) - M_i(t', u)| &\leq \sup_{\theta \in \Theta} \frac{|g(\tilde{Z}_i(u), \theta) - g(\tilde{Z}_i(u), \theta')|}{|\theta - \theta'|_1} \cdot \{|\theta(t) - \theta(t')|_1 + |d_u(t) - d_u(t')|_1\},
\end{aligned}$$

As long as $|t - u| < 1$ and $|t - t'|$ is small enough, we obtain $|t' - u| \leq 1$. So in the case that either $|t - u| < 1$ or $|t' - u| < 1$, Lipschitz continuity of $\theta(\cdot), \theta'(\cdot)$ implies that there exists some constant $\tilde{C} > 0$ such that $|d_u(t) - d_u(t')|_1 \leq \tilde{C}|t - t'|$, $|\theta(t) - \theta(t')|_1 \leq \tilde{C}|t - t'|$, $|d_u(t)|_1 \leq \tilde{C}$.

This implies

$$\begin{aligned}
|M_i^{(2)}(t, u) - M_i^{(2)}(t', u)| \quad \leq\quad &\tilde{C}b_n^{-1} L_{\hat{K}} \sup_{\theta \in \Theta} |g(\tilde{Z}_i(u), \theta)| \cdot |t - t'| \\
&+ 2|\hat{K}|_\infty \tilde{C}^2 \cdot \sup_{\theta \in \Theta} \frac{|g(\tilde{Z}_i(u), \theta) - g(\tilde{Z}_i(u), \theta')|}{|\theta - \theta'|_1} |t - t'| \\
&+ |\hat{K}|_\infty \tilde{C} \cdot \sup_{\theta \in \Theta} |g(\tilde{Z}_i(u), \theta)| \cdot |t - t'|.
\end{aligned}$$

(8.69)

With Lemma 8.3(i) we obtain the result.

Suppose now that Assumption 7.3 holds. As long as $|t - t'|$ is small enough and n is large enough, $|u - t| \leq b_n$ (or $|u - t'| \leq b_n$) and the twice differentiability of $\theta(\cdot)$ imply that $\sup_{\nu \in [0,1]} |\theta(u) - (\theta(t) + \nu d_u(t))|_1 < \iota$, $\sup_{\nu \in [0,1]} |\theta(u) - (\theta(t') + \nu d_u(t'))|_1 < \iota$. We then obtain

$$|M_i(t, u)| \leq \sup_{|\theta - \theta(u)|_1 < \iota} |\tilde{g}_{\theta(u)}(\zeta_i, \tilde{X}_i(u), \theta)|,$$

$$|M_i(t, u) - M_i(t', u)| \leq \bar{C} \cdot \sup_{\theta \neq \theta', |\theta - \theta(u)|_1 < \iota, |\theta' - \theta(u)|_1 < \iota} \frac{|\tilde{g}_{\theta(u)}(\zeta_i, \tilde{X}_i(u), \theta) - \tilde{g}_{\theta(u)}(\zeta_i, \tilde{X}_i(u), \theta')|}{|\theta - \theta'|_1}$$
$$\times \{|\theta(t) - \theta(t')|_1 + |d_u(t) - d_u(t')|_1,$$

giving appropriate results for (8.69) and thus the assertion with Lemma 8.5. $\qquad \square$

LEMMA 8.14. *Let* $U_{i,n}(t) := K_{b_n}(i/n - t) \cdot (1, (i/n - t)b_n^{-1})^{\mathsf{T}}$. *Let Assumption 7.1 or 7.3 hold with some* $r = 2 + \varsigma$, $\varsigma > 0$. *Then it holds that*

$$\sup_{t \in (0,1)} \big| \nabla_\eta \hat{L}_{n,b_n}^\circ(t, \eta_{b_n}(t)) - \mathbb{E} \nabla_\eta \hat{L}_{n,b_n}^\circ(t, \eta_{b_n}(t))$$

$$- (nb_n)^{-1} \sum_{i=1}^{n} U_{i,n}(t) \otimes \nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n)) \big| = O_{\mathbb{P}}(\beta_n b_n^2).$$

PROOF. Note that $\mathbb{E} \nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n)) = 0$ by Assumption 7.1(A1),(A3) or Assumption 7.3(A1'),(A3'). Put

$$\Pi_n(t)$$

$$:= (nb_n)^{-1} \sum_{i=1}^{n} U_{i,n}(t) \otimes \big\{ [\nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(t) + (i/n - t)\theta'(t)) - \nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n))]$$

$$- \mathbb{E}[\nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(t) + (i/n - t)\theta'(t)) - \nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n))] \big\}.$$

We have to prove that $\sup_{t \in \mathcal{T}_n} |\Pi_n(t)| = O_{\mathbb{P}}(\delta_n b_n^2)$. Define $M_i(t, u) := \int_0^1 \nabla_\theta^2 \ell(\tilde{Z}_i(u), \theta(t) + s(\theta(u) - \theta(t) - (u - t)\theta'(t)))ds$ and $M_i^{(2)}(t, u) = U_{i,n}(t) \otimes \big\{ M_i(t, u)\{\theta(u) - \theta(t) - (u - t)\theta'(t)\} \big\}$. By a Taylor expansion of $\nabla_\theta \ell$ w.r.t. θ, we have

$$\Pi_n(t) = (nb_n)^{-1} \sum_{i=1}^{n} (M_i^{(2)}(t, i/n) - \mathbb{E} M_i^{(2)}(t, i/n)).$$

We now apply a similar technique as in the proof of Lemma 8.10(iii), namely we use a chaining argument similar to (8.56) to prove

$$\mathbb{P}\big(\sup_{t \in (0,1)} |\Pi_n(t)| > Q \beta_n b_n^2 \big) \to 0,$$

for some $Q > 0$ large enough. Define the discretization $\mathcal{T}_{n,r} := \{l/r : l = 1, \ldots, r\}$ with $r = n^5$. By Lemma 8.13, we have with Markov's inequality for $Q > 0$:

$$\mathbb{P}\Big(\sup_{|t-t'|\leq r^{-1}} |\Pi_n(t) - \Pi_n(t')| > Q\beta_n b_n^2/2\Big) = O\Big(\frac{b_n^{-2}r^{-1}}{\beta_n b_n^2}\Big),$$

which converges to 0. Choose $\alpha = 1/2$. By Lemma 8.7(iii) or Lemma 8.8(iii) applied with $q = 2 + s$ (s small enough), we obtain that $\sup_u \Delta_{2+s}^{\sup_t |M^{(2)}(t,u)|}(k) = O(k^{-(1+\gamma)})$. Thus

$$\tilde{W}_{2+s,\alpha} := \sup_{u\in[0,1]} \sup_{t\in[0,1]} \|\sup_{t,\eta} |M_i^{(2)}(t,u)|\|_{2+\varsigma,\alpha} = \sup_{m\geq 0}(m+1)^\alpha \Delta_{2+s}^{\sup_t |M^{(2)}(t,u)|}(m)$$

$$(8.70) \qquad = O(b_n^2)$$

(the constant being independent of n) and

$$\tilde{W}_{2,\alpha} := \sup_{t,u} \|M_i^{(2)}(t,u)\|_{2,\alpha} = \sup_{m\geq 0}(m+1)^\alpha \sup_{u\in[0,1]} \sup_t \Delta_2^{M^{(2)}(t,u)}(m)$$

$$(8.71) \qquad = O(b_n^2)$$

(the constant being independent of n). We now apply Theorem 6.2 of [57] (the proof therein also works for the uniform functional dependence measure) with $q = 2 + s$, $\alpha = 1/2$ to $(M_i^{(2)}(t,i/n))_{t\in\mathcal{T}_{n,r}}$, where $l = 1 \vee \#(\mathcal{T}_{n,r}) \leq 5\log(n)$. For Q large enough, we obtain with some constant $C_{\alpha,s} > 0$:

$$\mathbb{P}\Big(\sup_{t'\in\mathcal{T}_{n,r}} |\Pi_n(t')| \geq Q\beta_n b_n^2/2\Big)$$

$$\leq \frac{C_{\alpha,s}n \cdot l^{1+s/2}\tilde{W}_{2+s,\alpha}^{2+s}}{(Q/2)^{2+s}(\beta_n b_n^2(nb_n))^{2+s}} + C_{\alpha,s}\exp\Big(-\frac{C_{\alpha,s}(Q/2)^2(\beta_n b_n^2(nb_n))^2}{n\tilde{W}_{2,\alpha}^2}\Big)$$

$$\lesssim n^{-\varsigma/2} + \exp\Big(-\frac{(nb_n)b_n^{-1}\log(n)}{n}\Big)$$

$$\rightarrow 0,$$

which finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

8.2. *Proofs and Lemmas for the SCB.* From Lemma 1 in [59], we adopt the following result:

LEMMA 8.15. *Let* $F_n(t) = \sum_{i=1}^n \hat{K}_{b_n}(t_i - t)V_i$, *where* $V_i, i \in \mathbb{Z}$ *are i.i.d.* $N(0, I_{s\times s})$. $b_n \to 0$ *and* $nb_n/\log^2(n) \to \infty$. *Let* $m^* = 1/b_n$. *Then*

$$(8.72) \quad \lim_{n\to\infty} \mathbb{P}\Big(\frac{1}{\sigma_{\hat{K},0}\sqrt{nb_n}} \sup_{t\in\mathcal{T}_n} |F_n(t)| - B_{\hat{K}}(m^*) \leq \frac{u}{\sqrt{2\log(m^*)}}\Big) = \exp(-2\exp(-u)).$$

where $B_{\hat{K}}$ *is defined in (3.7).*

The following lemma is an analogue of Lemma 2 in [59]. Since we use other Gaussian approximation rates from Theorem 8.17, we shortly state the proof for completeness.

LEMMA 8.16. *Let the assumptions and notations from Theorem 8.17 hold. Define*

$$D_{\tilde{h}}(t) := (nb_n)^{-1} \sum_{i=1}^n \hat{K}_{b_n}(i/n - t)\tilde{h}_i(i/n).$$

Assume that $\Sigma_{\tilde{h}}(t)$ is Lipschitz-continuous and that its smallest eigenvalue is bounded away from 0 uniformly on $[0,1]$. Assume that $\log(n)^4\big(b_n n^{(2\gamma+\varsigma\gamma-\varsigma)/(\varsigma+4\gamma+2\gamma\varsigma)}\big)^{-1} \to 0$ and $b_n \log(n)^{3/2} \to 0$. Then

$$(8.73) \quad \lim_{n\to\infty} \mathbb{P}\Big(\frac{\sqrt{nb_n}}{\sigma_{\hat{K},0}} \sup_{t\in\mathcal{T}_n} \Big|\Sigma_{\tilde{h}}^{-1}(t)D_{\tilde{h}}(t)\Big| - B_{\hat{K}}(m^*) \le \frac{u}{\sqrt{2\log(m^*)}}\Big) = \exp(-2\exp(-u)),$$

PROOF OF LEMMA 8.16. By Theorem 8.17 and summation-by-parts, there exist i.i.d. $V_i \sim N(0, I_{s\times s})$ such that
(8.74)

$$\sup_{t\in(0,1)} |D_{\tilde{h}}(t) - \Xi(t)| = O_{\mathbb{P}}\Big(\frac{n^{\frac{2\varsigma+2\gamma+\gamma\varsigma}{2\varsigma+8\gamma+4\gamma\varsigma}} \log(n)^{\frac{2\gamma(3+\varsigma)}{\varsigma+4\gamma+2\gamma\varsigma}}}{nb_n}\Big) = O_{\mathbb{P}}\Big(\frac{\log(n)^2 \big(b_n n^{\frac{2\gamma+\varsigma\gamma-\varsigma}{\varsigma+4\gamma+2\gamma\varsigma}}\big)^{-1/2}}{(nb_n)^{1/2}\log(n)^{1/2}}\Big),$$

where $\Xi(t) = (nb_n)^{-1} \sum_{i=1}^n \hat{K}_{b_n}(i/n-t)\Sigma_{\tilde{h}}(i/n)V_i$. Here, (8.74) is $o_{\mathbb{P}}((nb_n)^{-1/2}\log(n)^{-1/2})$ due to

$$\log(n)^4\big(b_n n^{(2\gamma+\varsigma\gamma-\varsigma)/(\varsigma+4\gamma+2\gamma\varsigma)}\big)^{-1} \to 0.$$

Since $\Sigma_{\tilde{h}}(\cdot)$ is Lipschitz continuous by Assumption (b), we can use a standard chaining argument in t (as it was done in Lemma 8.14 for $\Pi_n(t)$) and the fact that $(nb_n)^{-1}\sum_{i=1}^n(\Sigma_{\tilde{h}}(i/n) - \Sigma_{\tilde{h}}(t))\hat{K}_{b_n}(i/n - t)V_i \sim N(0, v_n)$, with $|v_n|_\infty \le C\frac{b_n}{n}$ for some constant $C > 0$ to obtain

$$\sup_{t\in(0,1)} |\Xi(t) - (nb_n)^{-1}\Sigma_{\tilde{h}}(t)\sum_{i=1}^n \hat{K}_{b_n}(i/n - t)V_i|$$

$$= \sup_{t\in(0,1)} \big|(nb_n)^{-1}\sum_{i=1}^n \hat{K}_{b_n}(i/n - t)(\Sigma_{\tilde{h}}(i/n) - \Sigma_{\tilde{h}}(t))V_i\big|$$

$$(8.75) \qquad = O_{\mathbb{P}}\Big(\frac{b_n\log(n)}{(nb_n)^{1/2}}\Big) = O_{\mathbb{P}}\Big(\frac{b_n\log(n)^{3/2}}{(nb_n)^{1/2}\log(n)^{1/2}}\Big),$$

which is $o_{\mathbb{P}}((nb_n)^{-1/2}\log(n)^{-1/2})$ due to $b_n\log(n)^{3/2} \to 0$. So the result follows from Lemma 8.15 in view of (8.74) and (8.75). □

For the following results, let us assume that there exists some measurable function $\tilde{H}(\cdot,\cdot)$ such that for each $t \in [0,1]$, $\tilde{h}_i(t) = \tilde{H}(t, \mathcal{F}_i) \in \mathbb{R}^s$ is well-defined. Put $S_{\tilde{h}}(i) := \sum_{j=1}^i \tilde{h}_j(j/n)$.

THEOREM 8.17 (Theorem 1 and Corollary 2 from Wu and Zhou [53]).   *Assume that for each component $j = 1, \ldots, s$:*

(a) $\sup_{t \in [0,1]} \|\tilde{h}_0(t)_j\|_{2+\varsigma} < \infty$,

(b) $\sup_{t \neq t' \in [0,1]} \|\tilde{h}_0(t)_j - \tilde{h}_0(t')_j\|_2 / |t - t'| < \infty$,

(c) $\sup_{t \in [0,1]} \delta_{2+\varsigma}^{\tilde{h}(t)_j}(k) = O(k^{-(\gamma+1)})$ *with some* $\gamma \geq 1$.

for some $\varsigma \leq 2$. *Then on a richer probability space, there are i.i.d.* $V_1, V_2, \ldots \sim N(0, I_{s \times s})$ *and a process* $S_{\tilde{h}}^0(i) = \sum_{j=1}^{i} \Sigma_{\tilde{h}}(j/n) V_j$ *such that* $(S_{\tilde{h}}(i))_{i=1,\ldots,n} \overset{d}{=} (S_{\tilde{h}}^0(i))_{i=1,\ldots,n}$ *and*

$$\max_{i=1,\ldots,n} |S_{\tilde{h}}(i) - S_{\tilde{h}}^0(i)| = O_{\mathbb{P}}(\pi_n).$$

where

$$(8.76) \qquad \pi_n = n^{(2\varsigma + 2\gamma + \gamma\varsigma)/(2\varsigma + 8\gamma + 4\gamma\varsigma)} \log(n)^{2\gamma(3+\varsigma)/(\varsigma + 4\gamma + 2\gamma\varsigma)}$$

and

$$\Sigma_{\tilde{h}}(t) = \Big(\sum_{j \in \mathbb{Z}} \mathbb{E}[\tilde{h}_0(t) \tilde{h}_j(t)^\mathsf{T}] \Big)^{1/2}.$$

PROOF OF THEOREM 3.3. By Proposition 9.1. Assumption 2.1 implies Assumption 7.1, 7.2. By Proposition 9.2 or Assumption 2.2 implies Assumption 7.3, 7.4. The results now follow from Theorem 8.18. $\square$

THEOREM 8.18 (Simultaneous confidence bands for $\theta(\cdot)$ and $\theta'(\cdot)$).   *Let C be a fixed $k \times s$ matrix with rank $s \leq k$. Define* $\hat{\theta}_{b_n, C}(t) := C^\mathsf{T} \hat{\theta}_{b_n}(t)$, $\hat{\theta}'_{b_n, C}(t) := C^\mathsf{T} \widehat{\theta}'_{b_n}(t)$ *and* $\theta_C(t) := C^\mathsf{T} \theta(t)$, $A_C(t) := V(t)^{-1} C$, $\Sigma_C^2(t) := A_C^\mathsf{T}(t) \Lambda(t) A_C(t)$.

Let Assumption 7.1 be fulfilled with $r = 2 + \varsigma$ for some $\varsigma > 0$. Assume that, for $\alpha_{exp} = (2\gamma + \varsigma\gamma - \varsigma)/(\varsigma + 4\gamma + 2\gamma\varsigma)$, $\log(n)(b_n n^{\alpha_{exp}})^{-1} \to 0$.

(i) *If $n b_n^7 \log(n) \to 0$, then*

$$\lim_{n \to \infty} \mathbb{P}\Big(\frac{\sqrt{n b_n}}{\sigma_{K,0}} \sup_{t \in \mathcal{T}_n} \Big| \Sigma_C^{-1}(t) \Big\{ \hat{\theta}_{b_n, C}(t) - \theta_C(t) - b_n^2 \frac{\mu_{K,2}}{2} \theta_C''(t) \Big\} \Big|$$
$$(8.77) \qquad\qquad - B_K(m^*) \leq \frac{u}{\sqrt{2 \log(m^*)}} \Big) = \exp(-2 \exp(-u)),$$

(ii) *If additionally, Assumption 7.2 is fulfilled and $n b_n^9 \log(n) \to 0$, then with $\hat{K}(x) = K(x)x$,*

$$\lim_{n \to \infty} \mathbb{P}\Big(\frac{\sqrt{n b_n^3} \mu_{K,2}}{\sigma_{K,2}} \sup_{t \in \mathcal{T}_n} \Big| \Sigma_C^{-1}(t) \Big\{ \hat{\theta}'_{b_n, C}(t) - \theta_C'(t) - b_n^2 \frac{\mu_{\hat{K},4}}{2 \mu_{\hat{K},2}} C^\mathsf{T} bias(t) \Big\} \Big|$$
$$(8.78) \qquad\qquad - B_{\hat{K}}(m^*) \leq \frac{u}{\sqrt{2 \log(m^*)}} \Big) = \exp(-2 \exp(-u)),$$

where in both cases $\mathcal{T}_n = [b_n, 1 - b_n]$, $m^* = 1/b_n$ *and*

$$(8.79) \qquad B_K(m^*) = \sqrt{2\log(m^*)} + \frac{\log(C_K) + (s/2 - 1/2)\log(\log(m^*)) - \log(2)}{\sqrt{2\log(m^*)}},$$

with

$$C_K = \frac{\left\{ \int_{-1}^{1} |K'(u)|^2 du / \sigma_{K,0}^2 \pi \right\}^{1/2}}{\Gamma(s/2)}.$$

The results hold true if instead of Assumption 7.1, 7.2, Assumption 7.3, 7.4 with some $r > 2$ *is assumed.*

PROOF OF THEOREM 8.18. Let $\tilde{k}_i(t) := \nabla_\theta \ell(\tilde{Z}_i(t), \theta(t))$ and $\hat{K}(x) = K(x)$ or $\hat{K}(x) = K(x)x$, respectively. Define

$$\Omega_C(t) := (nb_n)^{-1} \sum_{i=1}^{n} \hat{K}_{b_n}(i/n - t) A_C(i/n)^{\mathsf{T}} \tilde{k}_i(i/n)$$

and $D_{\tilde{k}}(t) = (nb_n)^{-1} \sum_{i=1}^{n} \hat{K}_{b_n}(i/n - t) \tilde{k}_i(i/n)$. Similar to the discussion of $\Pi_n(t)$ in the proof of Lemma 8.14 (note that the rates in (8.70) and (8.71) then change to $O(b_n)$ instead of $O(b_n^2)$), we can show that

$$(8.80) \qquad \sup_{t \in (0,1)} |\Omega_C(t) - A_C(t)^{\mathsf{T}} \cdot D_{\tilde{k}}(t)| = O_{\mathbb{P}}(\beta_n b_n) = O_{\mathbb{P}}\Big(\frac{b_n^{1/2} \log(n)}{(nb_n)^{1/2} \log(n)^{1/2}} \Big),$$

which is $o_{\mathbb{P}}((nb_n)^{-1/2} \log(n)^{-1/2})$ since $b_n \log(n)^2 \to 0$.

$\tilde{h}_i(t) := A_C(t)^{\mathsf{T}} \tilde{k}_i(t)$ is a locally stationary process with long-run variance $\Sigma_{\tilde{h}}^2(t) = \Sigma_C^2(t)$. By the result of Lemma 8.16, we have that

$$(8.81) \quad \lim_{n \to \infty} \mathbb{P}\Big(\frac{\sqrt{nb_n}}{\sigma_{\hat{K},0}} \sup_{t \in \mathcal{T}_n} |\Sigma_C^{-1}(t)\Omega_C(t)| - B_{\hat{K}}(m^*) \le \frac{u}{\sqrt{2\log(m^*)}} \Big) = \exp(-2\exp(-u)).$$

(i) By Theorem 8.2(i), we have

$$\sup_{t \in \mathcal{T}_n} \big| V(t)\{\hat{\theta}_{b_n}(t) - \theta(t)\} - b_n^2 \frac{\mu_{K,2}}{2} V(t)\theta''(t) - D_{\tilde{k}}(t) \big|$$

$$= O_{\mathbb{P}}\big(b_n^3 + (nb_n)^{-1} b_n^{-1/2} \log(n)^{3/2} + (nb_n)^{-1/2} b_n \log(n) \big)$$

$$(8.82) \qquad = O_{\mathbb{P}}\Big(\frac{(nb_n^7 \log(n))^{1/2} + (nb_n^2 \log(n)^{-4})^{-1/2} + b_n \log(n)^{3/2}}{(nb_n)^{1/2} \log(n)^{1/2}} \Big),$$

which is $o_{\mathbb{P}}((nb_n)^{-1/2} \log(n)^{-1/2})$ since $nb_n^7 \log(n) \to 0$, $nb_n^2 \log(n)^{-4} \to \infty$ and $b_n \log(n)^2 \to 0$. Together with (8.80) and (8.81) (with $\hat{K} = K$), this implies (8.77).

(ii) By Theorem 8.2(ii), we have

$$\sup_{t\in\mathcal{T}_n} \left|\mu_{K,2}V(t)b_n\{\hat{\theta}'_{b_n}(t) - \theta'(t)\} - b_n^3\frac{\mu_{K,4}}{2}V(t)\mathrm{bias}(t) - D_{\tilde{k}}(t)\right|$$

$$= O_\mathbb{P}\big(b_n^4 + (nb_n)^{-1}b_n^{-1/2}\log(n)^{3/2} + (nb_n)^{-1/2}b_n\log(n)\big)$$

$$= o_\mathbb{P}((nb_n)^{-1/2}\log(n)^{-1/2}),$$

as above. Together with (8.80) and (8.81) (with $\hat{K}(x) = K(x)x$), this implies (8.78). $\quad\square$

8.3. *Proofs of Section 4.*

PROOF OF PROPOSITION 4.1. (i) Lemma 8.10(i),(iii), Lemma 8.11 and the notation therein applied to $g = \nabla_\theta^2\ell$ imply

$$\sup_{t\in\mathcal{T}_n} |\hat{\mu}_{K,0,b_n}(t)\hat{V}_{b_n}(t) - \hat{\mu}_{K,0,b_n}(t)V(t)|$$

$$\leq \sup_{t\in\mathcal{T}_n,\eta\in E_n} |G_n^c(t,\eta) - \hat{G}_n(t,\eta)| + \sup_{t\in\mathcal{T}_n,\eta\in E_n} |\hat{G}_n(t,\eta)|$$

$$+ \sup_{t\in\mathcal{T}_n,\eta\in E_n} |\mathbb{E}\hat{B}_n(t,\eta) - V^\circ(t,\eta)| + \sup_{t\in\mathcal{T}_n} |V^\circ(t,\hat{\eta}_{b_n}) - \hat{\mu}_{K,0,b_n}(t)V(t)|$$

$$(8.83) \quad = O_\mathbb{P}((nb_n)^{-1}) + o_\mathbb{P}(\beta_n) + O(b_n) + \sup_{t\in\mathcal{T}_n} |V^\circ(t,\hat{\eta}_{b_n}) - \hat{\mu}_{K,0,b_n}(t)V(t)|.$$

We obtain similar as in the proof of Theorem 8.2(i) ((8.30) therein) that

$$\sup_{t\in\mathcal{T}_n} |\hat{\eta}_{b_n}(t) - \eta_{b_n}(t)| = O_\mathbb{P}((nb_n)^{-1/2}\log(n) + (nb_n)^{-1} + \beta_nb_n^2 + b_n^2).$$

Since $\eta \mapsto V^\circ(t,\eta)$ is Lipschitz continuous by Lemma 8.3, the result follows from (8.83) and $b_n\log(n) \to 0$.

(ii) follows similarly due to $\nabla_\theta\ell \cdot \nabla_\theta\ell^\mathsf{T} \in \mathcal{H}(2M_y, 2M_y, \chi, \bar{\bar{C}})$ with some $\bar{\bar{C}} > 0$.

$\quad\square$

PROOF OF PROPOSITION 4.3. Similar as in the proof of Theorem 8.2(i) by now using the explicit result of Lemma 8.11(a) applied to $g = \ell$ (both for Assumption 7.1 and 7.3), we obtain

$$\sup_{t\in(0,1)} \sup_{\eta\in E_n} |L_{n,b_n}^\circ(t,\eta) - \tilde{L}_{b_n}^\circ(t,\eta)| = O_\mathbb{P}(\beta_n + (nb_n)^{-1}) + O(b_n),$$

where $\tilde{L}_{b_n}^\circ(t,\eta) = \int_{-t/b_n}^{(1-t)/b_n} K(x)L(t,\eta_1 + \eta_2 x)dx$. By optimality of $\hat{\eta}_{b_n}(t)$,

$$0 \leq L_{n,b_n}^\circ(t,\theta(t)) - L_{n,b_n}^\circ(t,\hat{\eta}_{b_n}(t))$$

$$\leq \tilde{L}_{b_n}^\circ(t,\theta(t)) - \tilde{L}_{b_n}^\circ(t,\hat{\eta}_{b_n}(t)) + 2\sup_{\eta\in E_n} |L_{n,b_n}^\circ(t,\eta) - \tilde{L}_{b_n}^\circ(t,\eta)|.$$

This implies

$$\min\Big\{\int_{-1}^{0} K(x)\big\{L(t,\hat{\theta}_{b_n}(t)+b_n\widehat{\theta'_{b_n}}(t)x)-L(t,\theta(t))\big\}dx,$$

$$(8.84)\quad \int_{0}^{1} K(x)\big\{L(t,\hat{\theta}_{b_n}(t)+b_n\widehat{\theta'_{b_n}}(t)x)-L(t,\theta(t))\big\}dx\Big\} \leq 2\sup_{\eta\in E_n}|L^{\circ}_{n,b_n}(t,\eta)-\tilde{L}^{\circ}_{b_n}(t,\eta)|.$$

Assume that for some $\iota > 0$, $\limsup_{n\to\infty}\sup_{t\in(0,1)}|\hat{\eta}_{b_n}(t)-(\theta(t)^{\mathsf{T}},0)^{\mathsf{T}}| \geq \iota$. Then there exists $t \in (0,1)$ such that either (c1)

$$|\hat{\theta}_{b_n}(t)-\theta(t)| \geq \frac{1}{2}|b_n\widehat{\theta'_{b_n}}(t)|$$

and thus $|\hat{\theta}_{b_n}(t)-\theta(t)| > \iota/3$, or (c2)

$$|\hat{\theta}_{b_n}(t)-\theta(t)| < \frac{1}{2}|b_n\widehat{\theta'_{b_n}}(t)|,$$

and thus $|b_n\widehat{\theta'_{b_n}}(t)| > 2\iota/3$.

In case (c1), we have $|\hat{\theta}_{b_n}(t)+b_n\widehat{\theta'_{b_n}}(t)x-\theta(t)| \geq |\hat{\theta}_{b_n}(t)-\theta(t)|-|x||b_n\widehat{\theta'_{b_n}}(t)| \geq \frac{\iota}{6}$ for $x \in [0,\frac{1}{4}]$, thus with some $c_0 > 0$,

$$\int_{0}^{1} K(x)\big\{L(t,\hat{\theta}_{b_n}(t)+b_n\widehat{\theta'_{b_n}}(t)x)-L(t,\theta(t))\big\}dx \geq \int_{0}^{1/4} K(x)\big\{L(t,\hat{\theta}_{b_n}(t)+b_n\widehat{\theta'_{b_n}}(t)x)-L(t,\theta(t))\big\}dx \geq c_0$$

since $\theta \mapsto L(t,\theta)$ is continuous and attains its unique minimum at $\theta = \theta(t)$.

In case (c2), we have $|\hat{\theta}_{b_n}(t)+b_n\widehat{\theta'_{b_n}}(t)x-\theta(t)| \geq |x||b_n\widehat{\theta'_{b_n}}(t)|-|\hat{\theta}_{b_n}(t)-\theta(t)| \geq \frac{\iota}{6}$ for $x \in [\frac{3}{4},1]$, thus with some $c_0 > 0$,

$$\int_{0}^{1} K(x)\big\{L(t,\hat{\theta}_{b_n}(t)+b_n\widehat{\theta'_{b_n}}(t)x)-L(t,\theta(t))\big\}dx \geq \int_{3/4}^{1} K(x)\big\{L(t,\hat{\theta}_{b_n}(t)+b_n\widehat{\theta'_{b_n}}(t)x)-L(t,\theta(t))\big\}dx \geq c_0.$$

In both cases, (8.84) becomes a contradiction. Therefore,

$$\sup_{t\in(0,1)}|\hat{\eta}_{b_n}(t)-\eta_{b_n}(t)| = o_{\mathbb{P}}(1).$$

Using summation-by-parts and Gaussian approximation similar to that presented in Theorem 8.17 for the process $\nabla_\theta\ell(\tilde{Z}_i(i/n),\theta(i/n))$, there exists i.i.d. $V_1, V_2, \ldots \sim N(0, I_{s\times s})$ on a richer probability space such that, for π_n as in (8.76)

$$(8.85)$$

$$\sup_{t\in(0,1)}\Big|(nb_n)^{-1}\sum_{i=1}^{n}K_{b_n}(i/n-t)(\nabla_\theta\ell(\tilde{Z}_i(i/n),\theta(i/n))-V_i)\}\Big| = O_{\mathbb{P}}((nb_n)^{-1}\pi_n) = O_{\mathbb{P}}((nb_n)^{-1/2}\log(n)).$$

Thus one can replace $\sup_{t \in \mathcal{T}_n}$ by $\sup_{t \in (0,1)}$ in (8.27). A careful examination of the rest of the proof of Theorem 8.2(i) (with Lemma 8.12(8.59) replaced by Lemma 8.12(8.60)) now yields the result

$$(8.86) \qquad \sup_{t \in (0,1)} \left| \tilde{V}^{\circ}_{b_n}(t) \cdot (\hat{\eta}_{b_n}(t) - \eta_{b_n}(t)) - \nabla_\eta L^{\circ,c}_{n,b_n}(t, \eta_{b_n}(t)) \right| = O_{\mathbb{P}}(\tau_n^{(1)}),$$

where (we shortly write $\hat{\mu}_{K,j}(t) = \hat{\mu}_{K,j,b_n}(t)$)

$$\tilde{V}^{\circ}_{b_n}(t) = \begin{pmatrix} \hat{\mu}_{K,0}(t) & \hat{\mu}_{K,1}(t) \\ \hat{\mu}_{K,1}(t) & \hat{\mu}_{K,2}(t) \end{pmatrix} \otimes V(t).$$

By Lemma 8.10(i), Lemma 8.12 and Lemma 8.14, we obtain furthermore with $U_{i,n}(t) = (K_{b_n}(i/n - t), K_{b_n}(i/n - t) \cdot (i/n - t)b_n^{-1})^{\mathsf{T}}$:

$$\sup_{t \in (0,1)} \left| \nabla_\eta L^{\circ,c}_{n,b_n}(t, \eta_{b_n}(t)) - b_n^2 \begin{pmatrix} \hat{\mu}_{K,2}(t) \\ \hat{\mu}_{K,3}(t) \end{pmatrix} \otimes [V(t)\theta''(t)] \right.$$

$$(8.87) \qquad \left. -(nb_n)^{-1} \sum_{i=1}^n U_{i,n}(t) \otimes \nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n)) \right| = O_{\mathbb{P}}(\beta_n b_n^2 + b_n^3 + (nb_n)^{-1}).$$

Recalling the proof of Lemma 8.16, (8.74) and (8.75) and the proof of Theorem 8.18, (8.80) we see that there exist i.i.d. $V_i \sim N(0, I_{s \times s})$ such that both for $\hat{K} = K$ and $\hat{K}(x) = K(x) \cdot x$,

$$\sup_{t \in (0,1)} \left| A_C(t)^{\mathsf{T}}(nb_n)^{-1} \sum_{i=1}^n \hat{K}_{b_n}(i/n - t) \nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n)) - \Sigma_C(t)(nb_n)^{-1} \sum_{i=1}^n \hat{K}_{b_n}(i/n - t) V_i \right|$$

$$= O_{\mathbb{P}}\left(\frac{\log(n)^2 \left(b_n n^{\frac{2\gamma+\varsigma\gamma-\varsigma}{\varsigma+4\gamma+2\gamma\varsigma}} \right)^{-1/2}}{(8.88)} + \frac{b_n \log(n)^{3/2}}{(nb_n)^{1/2} \log(n)^{1/2}} + \frac{b_n^{1/2} \log(n)}{(nb_n)^{1/2} \log(n)^{1/2}} \right) =: O_{\mathbb{P}}(w_n).$$

With (8.86) and

$$\tilde{V}^{\circ}_{b_n}(t)^{-1} = \begin{pmatrix} \hat{\mu}_{K,0}(t) & \hat{\mu}_{K,1}(t) \\ \hat{\mu}_{K,1}(t) & \hat{\mu}_{K,2}(t) \end{pmatrix}^{-1} \otimes V(t)^{-1}$$

$$= \frac{1}{\hat{\mu}_{K,2}(t) N^{(0)}_{b_n}(t)} \begin{pmatrix} \hat{\mu}_{K,2}(t) V(t)^{-1} & -\hat{\mu}_{K,1}(t) V(t)^{-1} \\ -\hat{\mu}_{K,1}(t) V(t)^{-1} & \hat{\mu}_{K,0}(t) V(t)^{-1} \end{pmatrix},$$

we obtain:

$$\sup_{t \in (0,1)} \left| N^{(0)}_{b_n}(t) \cdot \{\hat{\theta}_{b_n,C}(t) - \theta_C(t)\} \right.$$

$$\left. - \left[A_C(t)^{\mathsf{T}} \nabla_{\eta_1} L^{\circ,c}_{n,b_n}(t, \eta_{b_n}(t)) - \frac{\hat{\mu}_{K,1}(t)}{\hat{\mu}_{K,2}(t)} A_C(t)^{\mathsf{T}} \nabla_{\eta_2} L^{\circ,c}_{n,b_n}(t, \eta_{b_n}(t)) \right] \right| = O_{\mathbb{P}}(\tau_n^{(1)}).$$

With (8.87) and (8.88), we have

$$\sup_{t\in(0,1)} \Big| N_{b_n}^{(0)}(t) \cdot \{\hat{\theta}_{b_n,C}(t) - \theta_C(t)\}$$

$$+ b_n^2 N_{b_n}^{(1)}(t)\theta_C''(t) - \Sigma_C(t)\Big\{Q_{b_n}^{(0)}(t) - \frac{\hat{\mu}_{K,1}(t)}{\hat{\mu}_{K,2}(t)}Q_{b_n}^{(1)}(t)\Big\}\Big|$$

$$= O_{\mathbb{P}}(\tau_n^{(1)} + (\beta_n b_n^2 + b_n^3 + (nb_n)^{-1}) + w_n),$$

which finishes the proof. $\qquad\square$

9. Proofs of Assumption sets.

9.1. *Case 1: Recursively defined models.*

PROPOSITION 9.1. *If Assumption 2.1 holds, then Assumption 7.1, 7.2 is fulfilled with every $r = 2 + \tilde{a}$, $\tilde{a} < a$, the corresponding M and $\gamma > 2$ arbitrarily large.*

It holds that $V(t) = \Lambda(t)$. If (i) $\mathbb{E}\zeta_0^3 = 0$, or (ii) $\mu(x,\theta) \equiv 0$ or (iii) $\sigma(x,\theta) \equiv \beta_0$ and $\mathbb{E}m(\tilde{X}_0(t)) = 0$, then

$$I(t) = \begin{pmatrix} I_k & 0 \\ 0 & (\mathbb{E}\zeta_0^4 - 1)I_{l+1}/2 \end{pmatrix} \cdot V(t),$$

where I_d denotes the d-dimensional identity matrix.

PROOF OF PROPOSITION 9.1. Choose $0 < \tilde{a} < a$ small enough such that (2.5) holds with $\|\zeta_0\|_{(2+\tilde{a})M}$ replaced by $\|\zeta_0\|_{2M}$ (this is possible due to continuity of the term in $\tilde{a} = 0$). Let $q = (2 + \tilde{a})M$. Let $\nu = (\nu_0, \ldots, \nu_l)^{\mathsf{T}}$ and $m = (m_1, \ldots, m_k)^{\mathsf{T}}$. As known from Proposition 4.3 and Lemma 4.4 in [13] the process $(Y_i)_{i=1,\ldots,n}$ described by (2.1) exists and fulfills Assumption 7.1(A5), (A7) with $\delta_q^Y(i) = O(c^i)$ for some $0 < c < 1$ and $q \geq 1$ if the recursion function $G_\zeta(y,t) := \mu(y,\theta(t)) + \sigma(y,\theta(t))\zeta$ obeys

$$(9.1) \qquad \Big\| \sup_{t\in[0,1]} \sup_{y\neq y'} \frac{|G_{\zeta_0}(y,t) - G_{\zeta_0}(y',t)|}{|y - y'|_{\chi,1}} \Big\|_q \leq 1$$

and

$$(9.2) \qquad \sup_{t\in[0,1]} \|C(\tilde{X}_t(t))\|_q < \infty, \quad C(y) := \sup_{t\neq t'} \frac{\|G_{\zeta_0}(y,t) - G_{\zeta_0}(y,t')\|_q}{|t - t'|},$$

where $|z|_{\chi,1} := \sum_{i=1}^p |z_i|\chi_i$ for some $\chi = (\chi_i)_{i=1,\ldots,p} \in \mathbb{R}_{\geq 0}^p$ with $|\chi|_1 = \sum_{i=1}^p \chi_i < 1$. Here, we can bound

$$(9.3) \qquad |\mu(y,\theta) - \mu(y',\theta)| \leq \sum_{i=1}^k |\alpha_i||y - y'|_{\kappa_{i\cdot},1} \leq |y - y'|_{\kappa^{(\mu)}(\alpha),1},$$

where $\chi^{(\mu)}(\alpha) := \sum_{i=1}^{k} |\alpha_i| \kappa_{i\cdot}$. Furthermore,

$$
\begin{aligned}
|\sigma(y,\theta)^2 - \sigma(y',\theta)^2| &\leq \sum_{i=0}^{l} \beta_i |\nu_i(y) - \nu_i(y')| \\
&\leq \sum_{i=0}^{l} \sqrt{\beta_i} |y - y'|_{\rho_{i\cdot},1} \cdot \left(\sqrt{\beta_i \nu_i(y)} + \sqrt{\beta_i \nu_i(y')} \right) \\
&\leq \sum_{i=0}^{l} \sqrt{\beta_i} |y - y'|_{\rho_{i\cdot},1} \cdot \left(\sigma(y,\theta) + \sigma(y',\theta) \right),
\end{aligned}
$$

i.e.

$$
(9.4) \qquad |\sigma(y,\theta) - \sigma(y',\theta)| \leq |y - y'|_{\chi^{(\sigma)}(\beta),1},
$$

where $\chi^{(\sigma)}(\beta) := \sum_{i=1}^{l} \sqrt{\beta_i} \rho_{i\cdot}$. Define

$$
\chi_j^{(\mu,max)} := \sup_t |\chi^{(\mu)}(\alpha(t))_j|, \qquad \chi_j^{(\sigma,max)} := \sup_t |\chi^{(\sigma)}(\beta(t))_j|.
$$

Since $\theta(t) = (\alpha(t)^{\mathsf{T}}, \beta(t)^{\mathsf{T}})^{\mathsf{T}} \in \Theta$, we have that

$$
\sum_{j=1}^{p} (\chi_j^{(\mu,max)} + \|\zeta_0\|_q \chi_j^{(\sigma,max)}) = \sum_{j=1}^{p} \left(\sup_t |\chi^{(\mu)}(\alpha(t))_j| + \|\zeta_0\|_q \sup_t |\chi^{(\sigma)}(\beta(t))_j| \right) < 1.
$$

Define $\chi_j := \chi_j^{(\mu,max)} + \|\zeta_0\|_q \chi_j^{(\sigma,max)}$. Then we have for all $t, y \neq y'$:

$$
|\mu(y,\theta(t)) - \mu(y',\theta(t))| + \|\zeta_0\|_q |\sigma(y,\theta(t)) - \sigma(y',\theta(t))| \leq |y - y'|_{\chi,1},
$$

which implies (9.1). Proposition 4.3 from [13] now implies the existence of Y_i, the stationary approximation $\tilde{Y}_i(t)$ and $\sup_t \|\tilde{Y}_0(t)\|_q < \infty$. By Lipschitz continuity of θ with constant L_θ, we have

$$
(9.5) \qquad |\mu(y,\theta(t)) - \mu(y,\theta(t'))| \leq L_\theta |t - t'| \sum_{i=1}^{k} |m_i(y)|,
$$

and

$$
\begin{aligned}
|\sigma(y,\theta(t))^2 - \sigma(y,\theta(t'))^2| &\leq L_\theta |t - t'| \sum_{i=0}^{l} \sqrt{\nu_i(y)} \frac{1}{2\beta_{min}^{1/2}} \left(\sqrt{\beta_i(t)\nu_i(y)} + \sqrt{\beta_i(t')\nu_i(y)} \right) \\
&\leq \frac{L_\theta}{2\beta_{min}^{1/2}} |t - t'| \sum_{i=0}^{l} \sqrt{\nu_i(y)} (\sigma(y,\theta(t)) + \sigma(y,\theta(t'))),
\end{aligned}
$$

which shows that

$$(9.6) \qquad |\sigma(y, \theta(t)) - \sigma(y, \theta(t'))| \leq \frac{L_\theta}{2\beta_{min}^{1/2}} \sum_{i=0}^{l} \sqrt{\nu_i(y)}.$$

Note that (2.4) implies

$$m_i(y), \sqrt{\nu_i(y)} \leq C_1|y|_1 + C_2,$$

with some constants $C_1, C_2 > 0$. By (9.5), (9.6), we have for $t \neq t'$

$$\begin{aligned} \|G_{\zeta_0}(y, t) - G_{\zeta_0}(y, t')\|_q &\leq |\mu(y, \theta(t)) - \mu(y, \theta(t'))| + \|\zeta_0\|_q|\sigma(y, \theta(t)) - \sigma(y, \theta(t'))| \\ &\leq C_3|t - t'|(1 + |y|_1), \end{aligned}$$

with some constant $C_3 > 0$. Since $\sup_t \|\tilde{Y}_0(t)\|_q < \infty$, (9.2) follows.

We now inspect the properties of the function ℓ. First note that the recursion of the stationary approximation,

$$\tilde{Y}_i(t) = \mu(\tilde{X}_i(t), \theta(t)) + \sigma(\tilde{X}_i(t), \theta(t))\zeta_i,$$

implies $\mathbb{E}\tilde{Y}_0(t) = 0$ and $\mathbb{E}\tilde{Y}_0(t)^2 = \mathbb{E}\mu(\tilde{X}_0(t), \theta(t))^2 + \mathbb{E}\sigma(\tilde{X}_0(t), \theta(t))^2 \geq \beta_{min}\nu_{min} > 0$. Furthermore, for $L(t, \theta) := \mathbb{E}\ell(\tilde{Z}_0(t), \theta)$ it holds that

$$\begin{aligned} L(t, \theta) - L(t, \theta(t)) &= \mathbb{E}\Big(\frac{\mu(\tilde{X}_0(t), \theta) - \mu(\tilde{X}_0(t), \theta(t))}{\sigma(\tilde{X}_0(t), \theta)}\Big)^2 \\ (9.7) \qquad &+ \mathbb{E}\Big[\frac{\sigma(\tilde{X}_0(t), \theta(t))^2}{\sigma(\tilde{X}_0(t), \theta)^2} - \log\frac{\sigma(\tilde{X}_0(t), \theta(t))^2}{\sigma(\tilde{X}_0(t), \theta)^2} - 1\Big]. \end{aligned}$$

In the following we use the notation $|x|_A^2 := x^\mathsf{T} Ax$ for a weighted vector norm. Note that

$$(9.8) \qquad \mathbb{E}\Big(\frac{\mu(\tilde{X}_0(t), \theta) - \mu(\tilde{X}_0(t), \theta(t))}{\sigma(\tilde{X}_0(t), \theta)}\Big)^2 \geq c_0|\alpha - \alpha(t)|_{M_1(t)}^2,$$

with $c_0 = (\max_{\theta \in \Theta} \max_i \theta_i^2)^{-1}$ and $M_1(t) := \mathbb{E}[\frac{m(\tilde{X}_0(t))m(\tilde{X}_0(t))^\mathsf{T}}{\mathbf{1}\nu(\tilde{X}(t))\nu(\tilde{X}(t))^\mathsf{T}\mathbf{1}}]$. If $M_1(t)$ was not positive definite, this would imply that there exists $v \in \mathbb{R}^k$ such that $v'M(t)v = 0$, which in turn would imply $v'\mu(\tilde{X}_0(t))\mu(\tilde{X}_0(t))v = 0$ a.s. and thus non-positive definiteness of $\mathbb{E}[\mu(\tilde{X}_0(t))\mu(\tilde{X}_0(t))^\mathsf{T}]$ which is a contradiction to the assumption.

By a Taylor expansion of $f(x) = x - \log(x) - 1$, we obtain

$$\begin{aligned} &\mathbb{E}\Big[\frac{\sigma(\tilde{X}_0(t), \theta(t))^2}{\sigma(\tilde{X}_0(t), \theta)^2} - \log\frac{\sigma(\tilde{X}_0(t), \theta(t))^2}{\sigma(\tilde{X}_0(t), \theta)^2} - 1\Big] \\ &\geq \frac{1}{2}\mathbb{E}\Big[\frac{(\sigma(\tilde{X}_0(t), \theta)^2 - \sigma(\tilde{X}_0(t), \theta(t))^2)^2}{(\sigma(\tilde{X}_0(t), \theta)^2 - \sigma(\tilde{X}_0(t), \theta(t))^2)^2 + \sigma(\tilde{X}_0(t), \theta)^4}\Big] \\ (9.9) \qquad &\geq \frac{c_0}{10}|\beta - \beta(t)|_{M_2(t)}^2, \end{aligned}$$

where $M_2(t) = \mathbb{E}\big[\frac{\nu(\tilde{X}_0(t))\nu(\tilde{X}_0(t))^{\mathsf{T}}}{\mathbf{1}\nu(\tilde{X}(t))\nu(\tilde{X}(t))^{\mathsf{T}}\mathbf{1}}\big]$ is positive definite by assumption (use a similar argumentation as above). By (9.7), (9.8) and (9.9) we conclude that $\theta \mapsto L(t, \theta)$ is uniquely minimized in $\theta = \theta(t)$. This shows 7.1(A3).

Omitting the arguments $z = (y, x)$ and θ, we have

$$(9.10) \quad \ell = \frac{1}{2}\Big[\frac{(y - \langle \alpha, m \rangle)^2}{\langle \beta, \nu \rangle} + \log\langle \beta, \nu \rangle\Big],$$

$$\nabla_\theta \ell = -\frac{\nabla_\theta m}{\sigma}\Big(\frac{y - m}{\sigma}\Big) + \frac{\nabla_\theta(\sigma^2)}{2\sigma^2}\Big[1 - \Big(\frac{y - m}{\sigma}\Big)^2\Big]$$

$$(9.11) \quad = \begin{pmatrix} -\frac{m}{\sigma}\big(\frac{y - m}{\sigma}\big) \\ \frac{\nu}{2\sigma^2}\big[1 - \big(\frac{y - m}{\sigma}\big)^2\big] \end{pmatrix} = \begin{pmatrix} \frac{m}{\langle \beta, \nu \rangle}(y - \langle \alpha, m \rangle) \\ \frac{\nu}{2\langle \beta, \nu \rangle}\big(1 - \frac{(y - \langle \alpha, m \rangle)^2}{\langle \beta, \nu \rangle}\big) \end{pmatrix},$$

$$\nabla_\theta^2 \ell = \frac{\nabla_\theta m \nabla_\theta m^{\mathsf{T}}}{\sigma^2} + \Big(\frac{y - m}{\sigma}\Big) \cdot \Big[\frac{\nabla_\theta m \nabla_\theta(\sigma^2)^{\mathsf{T}} + \nabla_\theta(\sigma^2)\nabla_\theta m^{\mathsf{T}}}{\sigma^3} - \frac{\nabla_\theta^2 m}{\sigma}\Big]$$

$$+ \frac{\nabla_\theta^2(\sigma^2)}{2\sigma^2}\Big[1 - \Big(\frac{y - m}{\sigma}\Big)^2\Big] + \frac{\nabla_\theta(\sigma^2)\nabla_\theta(\sigma^2)^{\mathsf{T}}}{2\sigma^4}\Big[2\Big(\frac{y - m}{\sigma}\Big)^2 - 1\Big]$$

$$= \begin{pmatrix} \frac{mm^{\mathsf{T}}}{\sigma^2} & \frac{y - m}{\sigma^2} \cdot m\nu^{\mathsf{T}} \\ \frac{y - m}{\sigma^2} \cdot \nu m^{\mathsf{T}} & \frac{\nu\nu^{\mathsf{T}}}{2\sigma^4}\big[2\big(\frac{y - m}{\sigma}\big)^2 - 1\big] \end{pmatrix}$$

$$(9.12) \quad = \begin{pmatrix} \frac{mm^{\mathsf{T}}}{\langle \beta, \nu \rangle} & \frac{y - \langle \alpha, m \rangle}{\langle \beta, \nu \rangle^2} \cdot m\nu^{\mathsf{T}} \\ \frac{y - \langle \alpha, m \rangle}{\langle \beta, \nu \rangle^2} \cdot \nu m^{\mathsf{T}} & \frac{\nu\nu^{\mathsf{T}}}{2\langle \beta, \nu \rangle^2}\big[2\frac{(y - \langle \alpha, m \rangle)^2}{\langle \beta, \nu \rangle} - 1\big] \end{pmatrix}.$$

Since ζ_1 is independent of $\tilde{X}_0(t) \in \mathcal{F}_0$ and $\mathbb{E}\zeta_1 = 0$, $\mathbb{E}\zeta_1^2 = 1$, we conclude that

$$\mathbb{E}[\nabla_\theta \ell(\tilde{Z}_0(t), \theta(t))|\mathcal{F}_{t-1}] = \mathbb{E}\Big[-\frac{\mu(\tilde{X}_j(t), \theta(t))}{\sigma(\tilde{X}_0(t), \theta(t))}\zeta_0 + \frac{\nu(\tilde{X}_0(t), \theta(t))}{2\sigma(\tilde{X}_j(t), \theta(t))^2}(1 - \zeta_0^2)\big|\mathcal{F}_{t-1}\Big] = 0,$$

i.e. $\nabla_\theta \ell(\tilde{Z}_1(t), \theta(t))$ is a martingale difference sequence, showing that $V(t) = \Lambda(t)$. We furthermore have that (we omit the arguments $(\tilde{X}_0(t), \theta(t))$ of μ, σ in the following):

$$V(t) = \mathbb{E}\nabla_\theta^2 \ell(\tilde{Z}_0(t), \theta(t)) = \begin{pmatrix} \mathbb{E}\big[\frac{mm^{\mathsf{T}}}{\langle \beta, \nu \rangle}\big] & 0 \\ 0 & \mathbb{E}\big[\frac{\nu\nu^{\mathsf{T}}}{2\langle \beta, \nu \rangle^2}\big] \end{pmatrix}.$$

With a similar argumentation as above, we conclude that $V(t)$ is positive definite (which then implies by continuity that the smallest eigenvalue of $V(t)$ is bounded away from 0 uniformly in t). By the martingale difference property, $I(t) = \Lambda(t)$. Omitting the arguments

$(\tilde{X}_0(t), \theta(t))$,

$$
\begin{aligned}
I(t) &= \mathbb{E}[\nabla_\theta(\tilde{Z}_j(t), \theta(t))\nabla_\theta(\tilde{Z}_0(t), \theta(t))^{\mathsf{T}}] \\
&= \begin{pmatrix} \mathbb{E}\left[\frac{mm^{\mathsf{T}}}{\sigma^2}\right] & \mathbb{E}[\zeta_0^3] \cdot \mathbb{E}\left[\frac{m\nu^{\mathsf{T}}}{2\sigma^3}\right] \\ \mathbb{E}[\zeta_0^3] \cdot \mathbb{E}\left[\frac{\nu m^{\mathsf{T}}}{2\sigma^3}\right] & \frac{\mathbb{E}[\zeta_0^4]-1}{4} \cdot \mathbb{E}\left[\frac{\nu\nu^{\mathsf{T}}}{\sigma^4}\right] \end{pmatrix} \\
&= \mathbb{E}\left[\frac{1}{\sigma^2}\begin{pmatrix} m \\ \frac{\mathbb{E}[\zeta_0^3]}{2\sigma}\nu \end{pmatrix}^{\mathsf{T}}\begin{pmatrix} m \\ \frac{\mathbb{E}[\zeta_0^3]}{2\sigma}\nu \end{pmatrix}\right] + \begin{pmatrix} 0 & 0 \\ 0 & (\frac{\mathbb{E}[\zeta_0^4]-\mathbb{E}[\zeta_0^3]^2-1}{4})\mathbb{E}\left[\frac{\nu\nu^{\mathsf{T}}}{\sigma^4}\right] \end{pmatrix},
\end{aligned}
$$

which is positive semidefinite since $\mathbb{E}[\zeta_0^3] = \mathbb{E}[\zeta_0(\zeta_0^2 - 1)] \le \mathbb{E}[\zeta_0^2]^{1/2}\mathbb{E}[(\zeta_0^2 - 1)^2]^{1/2} = (\mathbb{E}[\zeta_0^4] - 1)^{1/2}$. Positive definiteness follows from the fact that $(v_1, v_2)^{\mathsf{T}}I(t)(v_1, v_2) = 0$ implies $\nu^{\mathsf{T}}v_2 = 0$ a.s. from the last summand and $v_1^{\mathsf{T}}m + \frac{\mathbb{E}[\zeta_0^3]}{2\sigma}v_2^{\mathsf{T}}\nu = 0$ a.s. from the first summand, i.e. $v_1^{\mathsf{T}}m = 0$ a.s. which leads to a contradiction to either the positive definiteness of $\mathbb{E}[\nu\nu^{\mathsf{T}}]$ or $\mathbb{E}[mm^{\mathsf{T}}]$. So we obtain that Assumption 7.1(A4) is fulfilled.

A careful inspection of (9.10), (9.11) and (9.12) shows that $\ell, \nabla_\theta\ell, \nabla_\theta^2\ell \in \mathcal{H}(2, 3, \tilde{\chi}, \tilde{C})$ with some $\tilde{C} > 0$ and $\tilde{\chi} = (1, \ldots, 1, 0, 0, \ldots)$ consisting of $\max\{k, l\}$ ones followed by zeros, which shows Assumption 7.1(A1). In the special case $\mu(x, \theta) \equiv 0$, it seems as if no direct improvement of the value M is possible. In the special case of $\sigma(x, \theta)^2 \equiv \beta_0$, we have

$$
\begin{aligned}
\ell &= \frac{1}{2}\left[\frac{(y - \langle\alpha, m\rangle)^2}{\beta_0} + \log\beta_0\right], \\
\nabla_\theta\ell &= \begin{pmatrix} \frac{m}{\beta_0}(y - \langle\alpha, m\rangle) \\ \frac{1}{2\beta_0}(1 - \frac{(y-\langle\alpha,m\rangle)^2}{\beta_0}) \end{pmatrix}, \\
\nabla_\theta^2\ell &= \begin{pmatrix} \frac{mm^{\mathsf{T}}}{\beta_0} & \frac{y-\langle\alpha,m\rangle}{\beta_0^2}m \\ \frac{y-\langle\alpha,m\rangle}{\beta_0^2}m^{\mathsf{T}} & \frac{1}{2\beta_0^2}\left[2\frac{(y-\langle\alpha,m\rangle)^2}{\beta_0} - 1\right] \end{pmatrix},
\end{aligned}
$$

which implies that $\ell, \nabla_\theta\ell, \nabla_\theta^2\ell \in \mathcal{H}(2, 2, \tilde{\chi}, \tilde{C})$.

Now suppose that Assumption 7.2(B1) is fulfilled. We use results from Section 4 in [13] to show that the first derivative process $\partial_t\tilde{Y}_i(t)$ exists and fulfills a Lipschitz condition. By assumption, with some constant $C > 0$,

$$
\begin{aligned}
&|\partial_{x_j}G_{\zeta_0}(x, t) - \partial_{x_j}G_{\zeta_0}(x', t)| \\
\le\ &|\langle\alpha(t), \partial_{x_j}m(x) - \partial_{x_j}m(x')\rangle| \\
&\quad + \left|\frac{\langle\beta(t), \partial_{x_j}\nu(x)\rangle}{2\langle\beta(t), \nu(x)\rangle^{1/2}} - \frac{\langle\beta(t), \partial_{x_j}\nu(x')\rangle}{2\langle\beta(t), \nu(x')\rangle^{1/2}}\right| \cdot |\zeta_0| \\
\le\ &|\alpha(t)|_\infty C|x - x'|_1 \\
&\quad + |\zeta_0| \cdot \left(\frac{1}{2\beta_{min}^{1/2}}|\langle\beta(t), \partial_{x_j}\nu(x) - \partial_{x_j}\nu(x')\rangle|\right. \\
&\quad + \frac{|\langle\beta(t), \partial_{x_j}\nu(x')\rangle|}{2} \left.\frac{|\langle\beta(t), \nu(x) - \nu(x')\rangle|}{\langle\beta(t), \nu(x)\rangle^{1/2}\langle\beta(t), \nu(x')\rangle^{1/2}(\langle\beta(t), \nu(x)\rangle^{1/2} + \langle\beta(t), \nu(x')\rangle^{1/2})}\right).
\end{aligned}
$$

By assumption,
$$|\langle \beta(t), \partial_{x_j}\nu(x) - \partial_{x_j}\nu(x') \rangle| \leq C|\beta(t)|_\infty |x - x'|_1.$$

Furthermore,

$$
\begin{aligned}
|\langle \beta(t), \nu(x) - \nu(x') \rangle| &\leq \sum_{i=1}^{l} \beta_i(t) |\sqrt{\nu_i(x)} - \sqrt{\nu_i(x')}| \cdot |\sqrt{\nu_i(x)} + \sqrt{\nu_i(x')}| \\
&\leq |\beta(t)|_\infty |x - x'|_1 \sum_{i=1}^{l} \left(\sqrt{\nu_i(x)} + \sqrt{\nu_i(x')} \right).
\end{aligned}
$$

Since each component of $\beta(t)$ is lower bounded by β_{min} and therefore $\langle \beta(t), \nu(x) \rangle^{1/2} \geq \beta_{min}\sqrt{\nu_i(x)}$ for each i, we conclude that

$$\frac{|\langle \beta(t), \nu(x) - \nu(x') \rangle|}{\langle \beta(t), \nu(x) \rangle^{1/2}(\langle \beta(t), \nu(x) \rangle^{1/2} + \langle \beta(t), \nu(x') \rangle^{1/2})} \leq C|x - x'|_1,$$

with some constant $C > 0$. Finally, let e_j be the j-th unit vector in $\mathbb{R}^k$. Notice that for all i,

$$\frac{|\partial_{x_j}\nu_i(x)|}{\nu_i(x)^{1/2}} = 2|\partial_{x_j}\sqrt{\nu_i(x)}| \leq 2\lim_{h \to 0} \frac{|\sqrt{\nu_i(x)} - \sqrt{\nu_i(x + he_j)}|}{|h|} \leq \lim_{h \to 0} \frac{|he_j|_1}{|h|} \leq 1.$$

This shows that $\frac{|\langle \beta(t), \partial_{x_j}\nu(x') \rangle|}{\langle \beta(t), \nu(x') \rangle^{1/2}}$ is bounded and we obtain that for some constant > 0,

$$|\partial_{x_j}G_{\zeta_0}(x, t) - \partial_{x_j}G_{\zeta_0}(x', t)| \leq C(1 + |\zeta_0|)|x - x'|_1.$$

With similar but simpler arguments we obtain that

$$|\partial_t G_{\zeta_0}(x, t) - \partial_t G_{\zeta_0}(x', t)| \leq C(1 + |\zeta_0|)|x - x'|_1$$

and

$$\frac{|\partial_{x_j}G_{\zeta_0}(x, t) - \partial_{x_j}G_{\zeta_0}(x, t')|}{|t - t'| \cdot |x|_1} \leq C(1 + |\zeta_0|), \qquad \frac{|\partial_t G_{\zeta_0}(x, t) - \partial_t G_{\zeta_0}(x, t')|}{|t - t'| \cdot |x|_1} \leq C(1 + |\zeta_0|).$$

By Theorem 4.8 and Proposition 4.11 in [13], we obtain 7.2(B3) with $M_2 = 2M$.
Finally, straightforward calculations show that each component of $\partial_{x_j}\nabla_\theta^2\ell$ and $\nabla_\theta^3\ell$ is in $\mathcal{H}(M_2, M_2, \tilde{\chi}, \bar{C})$ with $\tilde{\chi} = (1, \ldots, 1, 0, \ldots, 0)$ consisting of p ones. This shows Assumption 7.2(B2).

$\square$

9.2. *Case 2: tvGARCH.*

PROPOSITION 9.2. *Let Assumption 2.2 hold. Then Assumption 7.3, 7.4 is fulfilled with every $r = 2 + \tilde{a}$, $\tilde{a} < \frac{a}{2}$, $M = 2$ and $\gamma > 2$ arbitrarily large. It holds that $\Lambda(t) = I(t) = ((\mathbb{E}\zeta_0^4 - 1)/2)V(t)$.*

PROOF OF PROPOSITION 9.2. We abbreviate $M_i(t) := M_i(\theta(t))$. Let $M = 1$. Fix $t \in [0, 1]$. Consider the recursion of the corresponding stationary approximation

$$
\begin{aligned}
\tilde{Y}_i(t) &= \tilde{\sigma}_i(t)^2 \zeta_i^2, \\
\tilde{\sigma}_i(t)^2 &= \alpha_0(t) + \sum_{j=1}^m \alpha_j(t)\tilde{Y}_{i-j}(t) + \sum_{j=1}^l \beta_j(t)\tilde{\sigma}_{i-j}(t)^2.
\end{aligned}
\tag{9.13}
$$

Define

$$
\begin{aligned}
\tilde{P}_i(t) &:= (\tilde{Y}_i(t), \ldots, \tilde{Y}_{i-m+1}(t), \tilde{\sigma}_i(t)^2, \ldots, \tilde{\sigma}_{i-l+1}(t)^2)^\mathsf{T}, \\
a_i(t) &:= (\alpha_0(t)\zeta_i^2, 0, \ldots, 0, \alpha_0(t), 0, \ldots, 0)^\mathsf{T}.
\end{aligned}
$$

For brevity, let $M_i(t) = M_i(\theta(t))$. Following Section 3.1 in [51], the model (9.13) admits the representation

$$
\tilde{P}_i(t) = M_i(t)\tilde{P}_{i-1}(t) + a_i(t).
\tag{9.14}
$$

Therefore, $\tilde{P}_i(t) = G_{\zeta_i}(\tilde{P}_{i-1}(t), t)$ with $G_{\zeta_i}(y, t) = M_i(t) \cdot y + a_i(t)$. Let $W_n(y, t) := G_{\zeta_n}(G_{\zeta_{n-1}}(...G_{\zeta_1}(y, t)...))$. Then we have

$$
W_n(y, t) - W_n(y', t) = M_n(t)M_{n-1}(t) \cdot ... \cdot M_1(t) \cdot (y - y').
$$

Note that $M_i(t)$ are i.i.d. matrices and for $i \geq 1$ they are also independent of y. Thus

$$
\mathbb{E}((W_n(y, t) - W_n(y', t))^{\otimes 2}) = \mathbb{E}(M_0(t)^{\otimes 2})^n \mathbb{E}((y - y')^{\otimes 2}).
$$

From the assumption $\rho(\mathbb{E}[M_0(\theta)^{\otimes 2}]) < 1$ and Theorem 2 in [52], we obtain existence and a.s. uniqueness of $\tilde{Y}_i(t) = H(t, \mathcal{F}_i)$, $\sup_{t \in [0,1]} \|\tilde{Y}_0(t)\|_q < \infty$ and $\sup_{t \in [0,1]} \delta_q^{\tilde{Y}(t)}(k) = \|\tilde{Y}_i(t) - \tilde{Y}_i(t)^*\|_q = O(c^k)$ for some $0 < c < 1$ and $q = 2 + \tilde{a}, \tilde{a} < a/2$. We can choose q to be slightly higher than 2 using continuity on moment index. This shows Assumption 7.3(A7').
(9.14) implies the explicit representation

$$
\tilde{P}_i(t) = \sum_{k=0}^\infty \Big(\prod_{j=0}^{k-1} M_{i-j}(t) \Big) a_{i-k}(t).
\tag{9.15}
$$

We therefore have for $t, t' \in [0,1]$:

$$
\begin{aligned}
\|\tilde{P}_i(t) - \tilde{P}_i(t')\|_q \;\le\; & \sum_{k=0}^{\infty} \sum_{l=0}^{k-1} \|\Big(\prod_{0 \le j < l} M_{i-j}(t) \Big)\|_q \|M_{i-l}(t) - M_{i-l}(t')\|_q \\
& \times \|\Big(\prod_{l < j \le k-1} M_{i-j}(t) \Big)\|_q \cdot \|a_{i-k}(t)\|_q \\
& + \sum_{k=0}^{\infty} \|\Big(\prod_{j=0}^{k-1} M_{i-j}(t) \Big)\|_q \|a_{i-k}(t) - a_{i-k}(t')\|_q.
\end{aligned}
$$

(9.16)

By Lipschitz continuity of $\theta(\cdot)$ we have

$$
\|a_0(t) - a_0(t')\|_q = |\alpha_0(t) - \alpha_0(t')|(\|\zeta_0^2\|_q, 0, \ldots, 0, 1, 0, \ldots, 0)^{\mathsf{T}} = O(|t - t'|)
$$

and

$$
\|M_0(t) - M_0(t')\|_q = (\|\zeta_0^2\|_q |f(\theta(t)) - f(\theta(t'))|, 0, \ldots, 0, |f(\theta(t)) - f(\theta(t'))|, 0, \ldots, 0)^{\mathsf{T}} = O(|t - t'|).
$$

Finally for the products of M_i matrices again note that they are independent and thus we can apply the same kronecker product technique along with the assumption $\rho(\mathbb{E}[M_0(\theta)^{\otimes 2}]) < 1$ as above and use continuity on the moment to deduce $\|\Big(\prod_{a \le j < b} M_{i-j}(t) \Big)\|_q \le c^{b-a+1}$ for some $0 < c < 1$. We conclude from the first component of (9.16), that for all $t, t' \in [0,1]$:

(9.17)
$$
\|\tilde{Y}_i(t) - \tilde{Y}_i(t')\|_q \le C \cdot |t - t'|,
$$

with some constant $C > 0$.

Put $P_i = (Y_i, \ldots, Y_{i-m+1}, \sigma_i^2, \ldots, \sigma_{i-l+1}^2)^{\mathsf{T}}$. Similarly to (9.14), we have

(9.18)
$$
P_i = M_i(i/n) P_{i-1} + a_i(i/n), \quad i = 1, \ldots, n.
$$

Note that i iterations of (9.18) lead to $P_0 = \tilde{P}_0(0)$, thus existence of Y_i follows from existence of $\tilde{Y}_i(0)$. We have

$$
\begin{aligned}
\|P_i - \tilde{P}_i(i/n)\|_q \;\le\; & \|M_i(i/n)\|_q \|P_{i-1} - \tilde{P}_{i-1}(i/n)\|_q \\
\le\; & \|M_0(i/n)\|_q \|P_{i-1} - \tilde{P}_{i-1}((i-1)/n)\|_q \\
& + \|M_0(i/n)\|_q \|\tilde{P}_0(i/n) - \tilde{P}_0((i-1)/n)\|_q.
\end{aligned}
$$

Iteration of this inequality leads to

$$
\|P_i - \tilde{P}_i(i/n)\|_q \le \sum_{k=1}^{i} \|\Big(\prod_{j=0}^{k} M_0((i-j)/n) \Big)\|_q \|\tilde{P}_0((i-k)/n) - \tilde{P}_0((i-k-1)/n)\|_q.
$$

For the product of M_i matrices the above technique is used. With that and (9.17), we conclude from the first component that $\|Y_i - \tilde{Y}_i(i/n)\|_q = O(n^{-1})$. This shows Assumption 7.3(A5').

Let $\Sigma(x,\theta) := (\sigma(x,\theta)^2, \ldots, \sigma(x_{(l-1)\to},\theta)^2)^{\mathsf{T}}$ and $A(x,\theta) := (\alpha_0 + \sum_{j=1}^m \alpha_j x_j, \ldots, \alpha_0 + \sum_{j=1}^m \alpha_j x_{j+l-1})^{\mathsf{T}}$, and

$$
B(\theta) = \begin{pmatrix} \beta_1 & \cdots & \cdots & \cdots & \beta_l \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.
$$

As said in Theorem 2.1 in [36], $\rho(\mathbb{E}M_0(\theta)^{\otimes 2}) < 1$ is a necessary and sufficient condition for the corresponding GARCH process with parameters θ to have 4-th moments. We conclude that $\rho(\mathbb{E}M_0(\theta)) < 1$ which by Proposition 1 in [19] implies $\rho(B(\theta)) < 1$. We have the explicit representation

$$
(9.19) \qquad \sigma(x,\theta)^2 = \sum_{k=0}^\infty \left(B(\theta)^k A(x_{k\to},\theta) \right)_1.
$$

Since $A(0,\theta) = (\alpha_0, 0, ..., 0)^{\mathsf{T}}$, we have

$$
\sigma(0,\theta)^2 = \alpha_0 \sum_{k=0}^\infty (B(\theta)^k)_{11}.
$$

From (9.19) we also obtain that

$$
(9.20) \qquad \sigma(x,\theta)^2 = c_0(\theta) + \sum_{j=1}^\infty c_j(\theta) \cdot x_j,
$$

where $c_j(\theta) \geq 0$ satisfies

$$
(9.21) \qquad \sup_{\theta \in \Theta} |c_j(\theta)| \leq C \cdot \rho^j
$$

with some $\rho \in (0,1)$ and $c_0(\theta) \geq \sigma_{min}^2 > 0$ (due to $\alpha_0 \geq \alpha_{min} > 0$). Due to the explicit representation (9.19) with geometrically decaying summands, it is easy to see that $\sigma(x,\theta)^2$ is four times continuously differentiable w.r.t. θ with

$$
(9.22) \qquad \nabla_\theta^k(\sigma(x,\theta)^2) = \nabla_\theta^k c_0(\theta) + \sum_{j=1}^\infty \nabla_\theta^k c_j(\theta) \cdot x_j, \qquad k \in \{0,1,2,3,4\},
$$

where $(\nabla_\theta^k c_j(\theta))_j$ is still geometrically decaying with $\sup_{\theta \in \Theta} |\nabla_\theta^k c_j(\theta)|_\infty \leq C \cdot \rho^j$, say.

From (9.22) we conclude that (component-wise) for $k = 0, 1, 2, 3$:

$$
\begin{aligned}
|\nabla_\theta^k(\sigma(x,\theta)^2) - \nabla_\theta^k(\sigma(x',\theta)^2)| &\leq C|x - x'|_{(\rho^j)_j, 1}, \\
|\nabla_\theta^k(\sigma(x,\theta)^2) - \nabla_\theta^k(\sigma(x,\theta')^2)| &\leq |\theta - \theta'|_1 \cdot \sup_{\theta \in \Theta} |\nabla_\theta^{k+1}(\sigma(x,\theta)^2)|_\infty \leq C|\theta - \theta'|_1 \cdot |x|_{(\rho^j)_j, 1}.
\end{aligned}
$$
(9.23)

We obtain that $\ell(y, x, \theta)$ is four times continuously differentiable and

$$
\begin{aligned}
\ell(y, x, \theta) &= \frac{1}{2}\Big(\frac{y}{\sigma(x,\theta)^2} + \log(\sigma(x,\theta)^2)\Big), \\
\nabla_\theta \ell(y, x, \theta) &= \frac{\nabla_\theta(\sigma(x,\theta)^2)}{2\sigma(x,\theta)^2}\Big(1 - \frac{y}{\sigma(x,\theta)^2}\Big), \\
\nabla_\theta^2 \ell(y, x, \theta) &= \Big[-\frac{\nabla_\theta(\sigma(x,\theta)^2)\nabla_\theta(\sigma(x,\theta)^2)^\mathsf{T}}{2\sigma(x,\theta)^4} + \frac{\nabla_\theta^2(\sigma(x,\theta)^2)}{2\sigma(x,\theta)^2}\Big]\Big(1 - \frac{y}{\sigma(x,\theta)^2}\Big) \\
&\quad + \frac{\nabla_\theta(\sigma(x,\theta)^2)\nabla_\theta(\sigma(x,\theta)^2)^\mathsf{T}}{2\sigma(x,\theta)^4} \cdot \frac{y}{\sigma(x,\theta)^2}.
\end{aligned}
$$

It was shown in the proof of Theorem 2.1 in [19], that $\theta \mapsto L(t, \theta) = \mathbb{E}\ell(\tilde{Z}_0(t), \theta)$ is uniquely minimized in $\theta = \theta(t)$, which shows Assumption 7.3(A3'). As in the proof of example Proposition 9.1, we obtain that

$$
V(t) = \mathbb{E}\Big[\frac{\nabla_\theta(\sigma(\tilde{X}_0(t), \theta(t))^2)\nabla_\theta(\sigma(\tilde{X}_0(t), \theta(t))^2)^\mathsf{T}}{2\sigma(\tilde{X}_0(t), \theta(t))^4}\Big] = I(t)\frac{2}{\mathbb{E}\zeta_0^4 - 1}.
$$

Furthermore,

$$
\nabla_\theta \ell(\tilde{Z}_i(t), \theta(t)) = \frac{\nabla_\theta(\sigma(\tilde{X}_i(t), \theta(t))^2)}{2\sigma(\tilde{X}_i(t), \theta(t))^2}\{1 - \zeta_i^2\},
$$

which shows that $\nabla_\theta \ell(\tilde{Z}_i(t), \theta(t))$ is a martingale difference sequence w.r.t. $\mathcal{F}_i$. Thus $\Lambda(t) = I(t)$. It was shown in the proof of Theorem 2.2 in [19] that $V(t)$ is positive definite for each $t \in [0, 1]$. By continuity, we conclude that Assumption 7.3(A4') is fulfilled.

Proof of Assumption 7.3(A1'): It holds that

$$
2|\ell(y, x, \theta) - \ell(y', x', \theta)| \leq |y - y'| \cdot \frac{1}{\sigma(x,\theta)^2} + |y'| \cdot \Big|\frac{1}{\sigma(x,\theta)^2} - \frac{1}{\sigma(x',\theta)^2}\Big| + |\log(\sigma(x,\theta)^2) - \log(\sigma(x',\theta)^2)|.
$$

Since $\sigma(x,\theta)^2 \geq \sigma_{min}^2 > 0$, Lipschitz continuity of $\log$ on $[\sigma_{min}, \infty)$ and (9.21), there exists some constant $C' > 0$ such that

$$
(9.24) \quad 2|\ell(y, x, \theta) - \ell(y', x', \theta)| \leq C'(|y - y'| + |x - x'|_{(\rho^j)_j, 1}) + |y'| \cdot \Big|\frac{1}{\sigma(x,\theta)^2} - \frac{1}{\sigma(x',\theta)^2}\Big|.
$$

Note that

$$\left|\frac{1}{\sigma(x,\theta)^2} - \frac{1}{\sigma(x',\theta)^2}\right| \le \frac{\sum_{j=0}^{\infty} c_j(\theta)|x_j.x_j'|}{\sigma(x,\theta)^2\sigma(x',\theta)^2} \le \sum_{j=1}^{\infty} \frac{c_j(\theta)|x_j - x_j'|}{(\sigma_{min}^2 + c_j(\theta)x_j)(\sigma_{min}^2 + c_j(\theta)x_j')}$$

$$\le \frac{1}{\sigma_{min}^2}\sum_{j=1}^{\infty} \frac{c_j(\theta)|x_j - x_j'|}{\sigma_{min}^2 + c_j(\theta)|x_j - x_j'|}.$$

The last step holds due to the following argument: It holds either $|x_j - x_j'| \le x_j$ or $|x_j - x_j'| \le x_j'$ since $x_j, x_j' \ge 0$. Therefore, one factor in the denominator can be lower bounded by σ_{min} and the other one by $\sigma_{min} + c_j(\theta)|x_j - x_j'|$. Following the ideas of [19], for arbitrarily small $s > 0$ we use the inequality $\frac{x}{1+x} \le x^s$ to obtain

$$\left|\frac{1}{\sigma(x,\theta)^2} - \frac{1}{\sigma(x',\theta)^2}\right| \le \frac{1}{\sigma_{min}^{4+2s}}\sum_{j=1}^{\infty} c_j(\theta)^s|x_j - x_j'|^s \le \frac{C}{\sigma_{min}^{4+2s}}|x - x|_{(\rho^{js})_j,s}$$

Together with (9.24), we obtain (7.6).

Using directly (9.24) and (9.23), we obtain

$$\sup_{\theta \in \Theta} \sup_{z \ne z'} \frac{|\ell(z,\theta) - \ell(z',\theta)|}{|z - z'|_{(\rho^j)_j,1} \cdot (1 + R_{2M-1,2M-1}(z) + R_{2M-1,2M-1}(z'))} < \infty.$$

Note that with some constant $C' > 0$,

$$2|\ell(z,\theta) - \ell(z,\theta')| \le |y| \cdot \left[\frac{1}{\sigma(x,\theta)^2} - \frac{1}{\sigma(x,\theta')^2}\right] + |\log(\sigma(x,\theta)^2) - \log(\sigma(x,\theta')^2)|$$

$$\le C'(1 + |y|) \cdot |\sigma(x,\theta)^2 - \sigma(x,\theta')^2|.$$

Together with (9.23), we obtain

$$\sup_{\theta \ne \theta'} \sup_z \frac{|\ell(z,\theta) - \ell(z,\theta')|}{|\theta - \theta'|_1 \cdot (1 + R_{2M,2M}(z) + R_{2M,2M}(z'))} < \infty.$$

This shows $\ell \in \mathcal{H}_0(2M, 2M, (\rho^j)_j, \bar{C})$ with some suitably chosen $\bar{C} > 0$.

Let $s > 0$ be arbitrary. It was shown in [19], (4.25) therein that with some small $\iota > 0$ only depending on s, Θ, it holds that

$$(9.25) \qquad \sup_{|\tilde\theta-\theta|<\iota} \frac{\sigma(x,\tilde\theta)^2}{\sigma(x,\theta)^2} \le \bar{C}(1 + |x|_{(\rho^{js})_j,s}^s).$$

Similarly, one can obtain for $k = 1, 2, 3$ that

$$(9.26) \qquad \sup_{|\tilde\theta-\theta|<\iota} \frac{\nabla_\theta^k\sigma(x,\tilde\theta)^2}{\sigma(x,\theta)^2} \le \bar{C}(1 + |x|_{(\rho^{js})_j,s}^s).$$

In the following we show that $\nabla_\theta \ell \in \mathcal{H}_s(2M, 2M, (\rho^j)_j, \bar{C})$ with some suitably chosen $\bar{C} > 0$. We have (component-wise):

$$
\begin{aligned}
&2|\nabla_\theta \ell(y, x, \theta) - \nabla_\theta \ell(y', x', \theta)| \\
\leq{}& |y - y'| \cdot \frac{1}{\sigma_{min}^2} \frac{|\nabla_\theta(\sigma(x,\theta)^2)|}{\sigma(x,\theta)^2} + |y'| \cdot \left| \frac{1}{\sigma(x,\theta)^2} - \frac{1}{\sigma(x',\theta)} \right| \cdot \frac{|\nabla_\theta(\sigma(x,\theta)^2)|}{\sigma(x,\theta)^2} \\
&+ \left(1 + \frac{|y'|}{\sigma_{min}^2} \right) \cdot \left(\frac{|\nabla_\theta(\sigma(x,\theta)^2) - \nabla_\theta(\sigma(x',\theta)^2)|}{\sigma_{min}^2} + \frac{|\nabla_\theta(\sigma(x',\theta)^2)|}{\sigma(x',\theta)^2 \sigma_{min}^2} |\sigma(x,\theta)^2 - \sigma(x,\theta')^2| \right).
\end{aligned}
$$

Using (9.23) and (9.26), we obtain (component-wise) with some suitably chosen $\bar{C} > 0$:
(9.27)
$$
2|\nabla_\theta \ell(y, x, \theta) - \nabla_\theta \ell(y', x', \theta)| \leq \bar{C} |z - z'|_{(\rho^j)_j, 1} \cdot (1 + R_{2M-1, 2M-1}(z) + R_{2M-1, 2M-1}(z'))^{1+s}.
$$

We have (component-wise):

$$
\begin{aligned}
&2|\nabla_\theta \ell(z, \theta) - \nabla_\theta \ell(z, \theta')| \\
\leq{}& |y| \cdot \left| \frac{1}{\sigma(x,\theta)^2} - \frac{1}{\sigma(x,\theta')} \right| \cdot \frac{|\nabla_\theta(\sigma(x,\theta)^2)|}{\sigma(x,\theta)^2} \\
&+ \left(1 + \frac{|y|}{\sigma_{min}^2} \right) \cdot \left(\frac{|\nabla_\theta(\sigma(x,\theta)^2) - \nabla_\theta(\sigma(x,\theta')^2)|}{\sigma_{min}^2} + \frac{|\nabla_\theta(\sigma(x,\theta')^2)|}{\sigma(x,\theta')^2 \sigma_{min}^2} |\sigma(x,\theta)^2 - \sigma(x,\theta')^2| \right).
\end{aligned}
$$

Using (9.23) and (9.26), we obtain (component-wise) with some suitably chosen $\bar{C} > 0$:

$$
(9.28) \qquad 2|\nabla_\theta \ell(z, \theta) - \nabla_\theta \ell(z, \theta')| \leq \bar{C} |\theta - \theta'|_1 \cdot (1 + R_{2M, 2M}(z) + R_{2M, 2M}(z'))^{1+s}.
$$

We conclude from (9.27) and (9.28) that $\nabla_\theta \ell \in \mathcal{H}_s(2M, 2M, (\rho^j)_j, \bar{C})$. The proof for $\nabla_\theta^2 \ell$ is similar in view of (9.23), (9.23) and (9.26) and therefore omitted.

Let $s > 0$ be arbitrary and $\iota > 0$ such that (9.25) and (9.26) hold. In the following we show that $\nabla_\theta \tilde{\ell} \in \mathcal{H}_{s,\iota}^{mult}(M, M, (\rho^j)_j, \bar{C})$ with some suitable chosen $\bar{C} > 0$. It holds that

$$
\nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y, x, \theta) = \frac{\nabla_\theta(\sigma(x,\theta)^2)}{2\sigma(x,\theta)^2} \left(1 - y \frac{\sigma(x,\tilde{\theta})^2}{\sigma(x,\theta)^2} \right).
$$

We have for $|\theta - \tilde{\theta}|_1 < \iota$:

$$
\begin{aligned}
&2|\nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y, x, \theta) - \nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y, x', \theta)| \\
\leq{}& |y| \cdot \left[\frac{|\sigma(x,\tilde{\theta})^2 - \sigma(x',\tilde{\theta})^2|}{\sigma_{min}^2} + \frac{\sigma(x',\tilde{\theta})^2}{\sigma(x',\theta)^2 \sigma_{min}^2} |\sigma(x,\theta)^2 - \sigma(x',\theta)^2| \right] \cdot \frac{|\nabla_\theta(\sigma(x,\theta)^2)|}{\sigma(x,\theta)^2} \\
&+ \left(1 + |y| \cdot \frac{\sigma(x',\tilde{\theta})^2}{\sigma(x',\theta)^2} \right) \cdot \left(\frac{|\nabla_\theta(\sigma(x,\theta)^2) - \nabla_\theta(\sigma(x',\theta)^2)|}{\sigma_{min}^2} + \frac{|\nabla_\theta(\sigma(x',\theta)^2)|}{\sigma(x',\theta)^2 \sigma_{min}^2} |\sigma(x,\theta)^2 - \sigma(x,\theta')^2| \right).
\end{aligned}
$$

Using (9.23) and (9.26), we obtain (component-wise) with some suitably chosen $\bar{C} > 0$:

$$(9.29) \qquad 2|\nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y, x, \theta) - \nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y, x', \theta)| \leq \bar{C}(1 + |y|) \cdot |x - x'|_{(\rho^j)_j, 1} \cdot |x|^s_{(\rho^{js})_j, s}.$$

We have for $|\theta - \tilde{\theta}|_1, |\theta' - \tilde{\theta}|_1 < \iota$:

$$2|\nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y, x, \theta) - \nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y, x, \theta')|$$
$$\leq |y| \cdot \frac{\sigma(x, \tilde{\theta})^2}{\sigma(x, \theta)^2 \sigma_{min}^2} |\sigma(x, \theta)^2 - \sigma(x, \theta')^2| \cdot \frac{|\nabla_\theta(\sigma(x, \theta)^2)|}{\sigma(x, \theta)^2}$$
$$+ \left(1 + |y| \cdot \frac{\sigma(x, \tilde{\theta})^2}{\sigma(x, \theta')^2}\right) \cdot \left(\frac{|\nabla_\theta(\sigma(x, \theta)^2) - \nabla_\theta(\sigma(x, \theta')^2)|}{\sigma_{min}^2} + \frac{|\nabla_\theta(\sigma(x, \theta')^2)|}{\sigma(x, \theta')^2 \sigma_{min}^2} |\sigma(x, \theta)^2 - \sigma(x, \theta')^2|\right).$$

Using (9.23) and (9.26), we obtain (component-wise) with some suitably chosen $\bar{C} > 0$:

$$(9.30) \qquad 2|\nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y, x, \theta) - \nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y, x, \theta')| \leq \bar{C}(1 + |y|) \cdot |\theta - \theta'|_1 \cdot R_{M,M}(1, x) \cdot |x|^s_{(\rho^{js})_j, s}.$$

We conclude from (9.29) and (9.30) that $\nabla_\theta \tilde{\ell} \in \mathcal{H}_{s,\iota}^{mult}(M, M, (\rho^j)_j, \bar{C})$. The proof for $\nabla_\theta^2 \tilde{\ell}$ is similar in view of (9.23), (9.23) and (9.26) and therefore omitted.

Regarding Assumption 7.2, notice that from the explicit representation uniformly in t we have that

$$\partial_t \tilde{P}_i(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{k-1} \left(\prod_{0 \leq j < l} M_{i-j}(t)\right) \partial_t M_{i-l}(t) \left(\prod_{l < j \leq k-1} M_{i-j}(t)\right) a_{i-k}(t)$$
$$+ \sum_{k=0}^{\infty} \left(\prod_{j=0}^{k-1} M_{i-j}(t)\right) \partial_t a_{i-k}(t)$$

exists a.s. and has q-th moments, so does its first component $\partial_t \tilde{Y}_t(t)$. Similar arguments that were used to prove (9.17) can be applied here and yield for $t, t' \in [0, 1]$:

$$\|\partial_t \tilde{Y}_i(t) - \partial_t \tilde{Y}_i(t')\|_q \leq C' \cdot |t - t'|,$$

with some constant $C' > 0$, i.e. Assumption 7.4(B3') is shown.

From (9.20) and $\sup_{\theta \in \Theta} \rho(B(\theta)) < 1$, it follows that $x_i \mapsto \sigma(x, \theta)^2$ is differentiable for all $i \in \mathbb{N}$ and

$$\partial_{x_i} \nabla_\theta^k(\sigma(x, \theta)^2) = \nabla_\theta^k c_i(\theta), \quad k = 0, 1, 2.$$

Let $M' = 1$. Similar as above it can be seen that $\nabla_\theta^3 \tilde{\ell} \in \mathcal{H}_{s,\iota}^{mult}(M', M', (\rho^j)_j, \bar{C})$. Note that

$$
\begin{aligned}
\partial_{x_i} \nabla_\theta^2 \tilde{\ell}_{\tilde{\theta}}(y, x, \theta) \;=\; & \Big[- \frac{\nabla_\theta c_i(\theta)}{\sigma(x,\theta)^2} \frac{\nabla_\theta(\sigma(x,\theta)^2)^\mathsf{T}}{2\sigma(x,\theta)^2} - \frac{\nabla_\theta(2\sigma(x,\theta)^2)}{\sigma(x,\theta)^2} \cdot \frac{\nabla_\theta c_i(\theta)^\mathsf{T}}{\sigma(x,\theta)^2} \\
& + \frac{c_i(\theta)}{\sigma(x,\theta)^2} \frac{\nabla_\theta(\sigma(x,\theta)^2)\nabla_\theta(\sigma(x,\theta)^2)^\mathsf{T}}{\sigma(x,\theta)^4} \\
& + \frac{\nabla_\theta^2 c_i(\theta)}{2\sigma(x,\theta)^2} - \frac{c_i(\theta)}{\sigma(x,\theta)^2} \frac{\nabla_\theta^2(\sigma(x,\theta)^2)}{2\sigma(x,\theta)^2} \Big] \cdot \Big(1 - y\frac{\sigma(x,\tilde{\theta})^2}{\sigma(x,\theta)^2}\Big) \\
& + \Big[- \frac{\nabla_\theta(\sigma(x,\theta)^2)\nabla_\theta(\sigma(x,\theta)^2)^\mathsf{T}}{2\sigma(x,\theta)^4} + \frac{\nabla_\theta^2(\sigma(x,\theta)^2)}{2\sigma(x,\theta)^2} \Big] \cdot \frac{c_i(\theta)}{\sigma(x,\theta)^2} \frac{\sigma(x,\tilde{\theta})^2}{\sigma(x,\theta)^2} \cdot y \\
& + \text{similar terms (derivative of second summand)}
\end{aligned}
$$

Each summand contains a factor $\nabla_\theta^k c_i(\theta)$ which is geometrically decaying by (9.26). Similar as above one can therefore see that $\nabla_\theta^2 \tilde{\ell}_{\tilde{\theta}} \in \mathcal{H}_{s,\iota}^{mult}(M', M', (\rho^j)_j, \bar{C}\rho^i)$. This shows Assumption 7.4(B2'). $\qquad\square$